Фундаментална матрица (ФМ)

$X'(t) = A(t)\,X(t)$

ОР: $X(t) = \Phi(t)\cdot c,\ c\in\mathbb{R}^n$ ($n\times n$)

$\hookrightarrow$ ФМ

$\Phi'(t) = A(t)\cdot\Phi(t)$

$\Phi(t) = [\varphi_1(t) \ \dots \ \varphi_n(t)]$ $\to$ $\varphi_1, \dots, \varphi_n$ решења система и

$\to$ они су лин. нез. $W(t)\neq 0$

$\det\Phi(t) = W(t)$ – Вронскијан

Пр. ЛСДЈКК $X' = AX$, једна ФМ је $\Phi(t) = e^{tA}$:

1) $\Phi'(t) = \frac{d}{dt}(e^{tA}) \overset{(5)}{=} Ae^{tA} = A\Phi(t)$

2) $W(t) = \det(e^{tA}) \overset{(6)}{=} e^{tr(tA)} \neq 0$

① $A\in M_2(\mathbb{R})$

$B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$e^{tA} = \begin{bmatrix} e^t & 0 \\ -3e^t+3e^{2t} & e^{2t} \end{bmatrix}$

а) Одредити једну ФМ $X' = \underbrace{B^{-1}AB}_{\in M_2(\mathbb{R})}X$.

б) Одредити $B^{-1}AB$.

а) $\Phi(t) = e^{tB^{-1}AB} \overset{(7)}{=} B^{-1}e^{tA}B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} e^{tA}B = \begin{bmatrix} 7e^t - 6e^{2t} & 14e^t - 14e^{2t} \\ -3e^t + 3e^{2t} & -6e^t + 7e^{2t} \end{bmatrix}$

б) $X' = CX$, $C = B^{-1}AB$

$e^{tC} = \Phi(t)$

I) $e^t, e^{2t} \rightsquigarrow C \sim \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \exists S\in GL_2(\mathbb{R}),$ $C = S\cdot\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\cdot S^{-1}$

$\Phi(t) = e^{tC} = S\cdot e^{t\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}\cdot S^{-1} = \dots$

$S = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \rightsquigarrow S^{-1} = \frac{1}{\alpha\delta - \beta\gamma}\begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}$

II) $\Phi'(t) = C \cdot \Phi(t)$

$W(t) \neq 0 \Rightarrow \exists \Phi^{-1}(t)$ $\Big\} \Rightarrow C = \Phi'(t) \cdot \Phi^{-1}(t) = \ldots$

решение: $C = \begin{bmatrix} -5 & -14 \\ 3 & 8 \end{bmatrix}$

Смене

② Решите систем:

$$\begin{cases} t^2 x_1 x_1' = \dfrac{x_1^2}{2} + e^{2x_2} \\ 2t^2 x_2' = -3\dfrac{x_1^2}{e^{2x_2}} - 4 \end{cases}$$

$y_1 = x_1^2 \longrightarrow y_1' = 2x_1x_1'$

$y_2 = e^{2x_2} \longrightarrow y_2' = 2e^{2x_2} x_2'$

$$\Rightarrow \begin{cases} t^2 \cdot \dfrac{y_1'}{2} = \dfrac{y_1}{2} + y_2 & / \cdot 2 \\ 2t^2 \cdot \dfrac{y_2'}{2e^{2x_2}} = -3\dfrac{y_1}{y_2} - 4 & / \cdot y_2 = e^{2x_2} \end{cases}$$

$t^2 \cdot y_1' = y_1 + 2y_2$

$t^2 \cdot y_2' = -3y_1 - 4y_2$

$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 \\ -3 & -4 \end{bmatrix}$

? $t^2 \cdot y' = Ay$

смена: $t \rightsquigarrow \tau$

$f(t) \cdot y' = Ay \xrightarrow{t \to \tau} y' = Ay$ ($\frac{d}{dt}$, $\frac{d}{d\tau}$)

$\frac{dy}{d\tau} = f(t) \cdot \frac{dy}{dt}$

условие: $f(t) = \frac{dt}{d\tau}$

$t^2 = \frac{dt}{d\tau} \Rightarrow d\tau = \frac{dt}{t^2} \Big/ \int$

$\tau = \int \frac{dt}{t^2} = -\frac{1}{t} \ (+c)$

$$\tau = \int \frac{dt}{t^2} = -\frac{1}{t} \ (+c)$$

$$\left.\begin{aligned} y_1'(\tau) &= \frac{dt}{d\tau} \cdot y_1'(t) = t^2 \cdot y_1'(t) \\ y_2'(\tau) &= \ldots \qquad\quad = t^2 \cdot y_2'(t) \end{aligned}\right\} \longrightarrow Y'(\tau) = A \cdot Y(\tau) \quad \text{шсдэКА}$$

$\vdots$

<u>системи $\leftrightarrow$ јне. виших реда</u>

(3) $\begin{cases} x_1' = p x_1 - q x_2 \\ x_2' = q x_1 + p x_2 \end{cases}$ (*) $\qquad p, q \in \mathbb{R} \setminus \{0\}$

2 јне 1. реда $\rightarrow$ 1 јна 2. реда

$$x_2 = \frac{p x_1 - x_1'}{q}$$

дома: решити (*) и (#) и упоредити

$$x_2' = \frac{p x_1' - x_1''}{q}$$

$$\frac{p x_1' - x_1''}{q} = q x_1 + p \cdot \frac{p x_1 - x_1'}{q} \quad / \cdot q$$

$$p x_1' - x_1'' = q^2 x_1 + p^2 x_1 - p x_1'$$

$$x_1'' - 2p x_1' + (p^2 + q^2) x_1 = 0 \quad (\#)$$

(4) $x''' - 2x'' + x = 0$ $\qquad$ 1 јна 3. реда $\rightarrow$ 3 јне 1. реда

$x_1 = x$

$x_2 = x' = x_1'$

$x_3 = x_2' = x_1'' = x''$

(x_1, x_2, x_3) - нове пром.

$$\left.\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= 2x_3 - x_1 \end{aligned}\right\}$$

$$x_3' = x''' = 2x'' - x = 2x_3 - x_1$$

5. $\left.\begin{array}{l} \boxed{x''' = ty' - \sin t \cdot x' + t} \\ \boxed{y'' = x'' - \cos(x' \cdot y)} \end{array}\right\}$ 3. реда + 2. реда = 5. реда = 5 јна 1. реда

$x_1 = x$

$\boxed{\begin{array}{l} x_2 = x' \\ x_3 = x'' \end{array}}$

$y_1 = y$

$\boxed{y_2 = y'}$

$x''' = x_3' = t y_2 - \sin t \cdot x_2 + t$

$y'' = y_2' = x_3 - \cos(x_2 \cdot y_1)$

систем,

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= t y_2 - \sin t \cdot x_2 + t \\ y_1' &= y_2 \\ y_2' &= x_3 - \cos(x_2 \cdot y_1) \end{aligned}$$

Специјално: $x^{(n)} + a_1 x^{(n-1)} + a_2 x^{(n-2)} + \dots + a_{n-1} x' + a_n x = 0$ (ЛЈ)

$\varphi : x \mapsto X$

$X' = AX$ (ЛС)

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\vdots \\ x_{n-1}' &= x_n \\ x_n' &= -a_n x_1 - a_{n-1} x_2 - \dots - a_1 x_n \end{aligned}$$

где је $A = \begin{bmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & \dots & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & \dots & & \dots & 1 \\ -a_n & -a_{n-1} & \dots & & \dots & -a_1 \end{bmatrix}$

Закључак.

1. $\mathcal{R}_{(ЛЈ)}$ – решења (ЛЈ) је векторски простор дим. n
   (φ изоморфизам)
   линеарно

2. Ако је λ корен полинома $\lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1}\lambda + a_n = 0$ (кп)
   (кар. полином ЛЈ)

сопст. вр. матрице A су нуле (кп)

онда

- $\lambda \in \mathbb{R}$ проста нула $\rightsquigarrow x(t) = e^{\lambda t}$
- $\lambda \in \mathbb{R}$ нула реда $k \rightsquigarrow x_1(t) = e^{\lambda t}, x_2(t) = te^{\lambda t}, \dots, x_k(t) = t^{k-1} e^{\lambda t}$
- $\lambda = \alpha \pm i\beta$ проста $\rightsquigarrow x(t) = e^{\alpha t} \cos \beta t$ и $y(t) = e^{\alpha t} \sin \beta t$
- $\lambda = \alpha \pm i\beta$ нула реда $k \rightsquigarrow \begin{cases} x_j(t) = t^j e^{\alpha t} \cos \beta t \\ y_j(t) = t^j e^{\alpha t} \cos \beta t \end{cases}$ $j = 0, \dots, k-1$

Овако добијемо базу простора $\mathcal{R}_{(ЛЈ)}$

Дн-1

Овако добијемо базу простора $\mathcal{R}_{(ЛЈ)}$

$$\det(A-\lambda E) = \det\left(\underbrace{\begin{bmatrix} -\lambda & 1 & 0 & \cdots & 0 \\ 0 & -\lambda & 1 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & 0 & \cdots & & 1 \\ -a_n & -a_{n-1} & \cdots & & -a_1-\lambda \end{bmatrix}}_{D_n}\right) = -\lambda \cdot \underbrace{\begin{vmatrix} -\lambda & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ -a_{n-1} & \cdots & & -a_1-\lambda \end{vmatrix}}_{D_{n-1}} + (-a_n)\cdot(-1)^{n+1}\cdot \begin{vmatrix} 1 & 0 & \cdots & 0 \\ -\lambda & 1 & \cdots & 0 \\ & \ddots & \ddots & \\ 0 & \cdots & & 1 \end{vmatrix} =$$

$1 \cdot 1 \cdots 1 = 1$

$$= -\lambda D_{n-1} + a_n(-1)^n \cdot 1 = -\lambda D_{n-1} + (-1)^n a_n$$

$$D_1 = -a_1 - \lambda$$

$$D_2 = a_1\lambda + \lambda^2 + a_2$$

$$D_3 = -a_1\lambda^2 - \lambda^3 - a_2\lambda - a_3$$

$$\vdots$$

$$D_n = (-1)^n \cdot \left(\lambda^n + a_1\lambda^{n-1} + \dots + a_n\right)$$

→ докажите индукцијом

$$\det(A-\lambda E) = 0 \iff \lambda^n + a_1\lambda^{n-1} + \dots + a_n = 0$$

<u>нпр.</u> $\lambda \in \mathbb{R}$ у систему: $x_1(t) = e^{\lambda t} \cdot v_1$, $\lambda v_1 = A v_1$

$$\left[x_1'(t) = \lambda e^{\lambda t} v_1 = A v_1 e^{\lambda t} = A x_1(t) \right]$$

→ свака врста је $\underline{\underline{e^{\lambda t} \cdot c_i}}$ (по трансф.) $\Rightarrow$ базни ел. за ЛДЈВРКК.

<u>ЛДЈВРКК</u>

$$x^{(n)} + a_{n-1}(t)\, x^{(n-1)} + \dots + a_1(t)\, x = f(t)$$

$f \equiv 0$ – хомогена

ОР: $x(t) = x_p(t) + x_H(t)$ → ОР хомогене

→ код сваке лин. јне

↳ партикуларно нехомогене

<u>хомогене</u>: ⑥ 2) $x''' - 13x' - 12x = 0 \longrightarrow \lambda^3 - 13\lambda - 12 = 0 \longrightarrow (\lambda+1)\underbrace{(\dots)}_{\lambda^2-\lambda-12} = 0$ $\Big\}$ $\lambda_1 = -1 \to e^{-t}$, $\lambda_2 = -3 \to e^{-3t}$, $\lambda_3 = 4 \to e^{4t}$

$$\text{ОР}: x(t) = c_1 e^{-t} + c_2 e^{-3t} + c_3 e^{4t}, \quad c_1, c_2, c_3 \in \mathbb{R}$$

б) $x''' - 7x'' + 16x' - 12x = 0$

$$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$$

$$(\lambda-2)(\lambda^2 - 5\lambda + 6) = 0 \longrightarrow \lambda_1 = \lambda_2 = 2, \quad \lambda_3 = 3$$

$$\text{OP}: \ x(t) = c_1 e^{2t} + c_2 t e^{2t} + c_3 e^{3t}, \quad c_1, c_2, c_3 \in \mathbb{R}$$

в) $x''' - 3x'' + 9x' + 13x = 0$

$\vdots$

$$\lambda_1 = -1 \longrightarrow e^{-t}$$

$$\lambda_{2/3} = 2 \pm 3i \longrightarrow e^{2t}\cos 3t, \ e^{2t}\sin 3t$$

г) $x^{(6)} - 4x^{(5)} + 8x^{(4)} - 8x''' + 4x'' = 0$

$$\lambda^6 - 4\lambda^5 + 8\lambda^4 - 8\lambda^3 + 4\lambda^2 = 0$$

$$\lambda^2 \underbrace{(\lambda^4 - 4\lambda^3 + 8\lambda^2 - 8\lambda + 4)} = 0$$

$$(\lambda^2 + a\lambda + b)(\lambda^2 + c\lambda + d) \quad = \quad (\lambda^2 - 2\lambda + 2)^2$$

$$a + c = -4$$
$$b + d + ac = 8$$
$$ad + bc = -8$$
$$bd = 4$$

пробамо: $b = d = 2 \ \text{---} \ a = c = -2$

$$\lambda_1 = \lambda_2 = 0 \rightarrow e^{0t}, te^{0t} \rightarrow 1, t$$

$$\lambda_{3/4} = \lambda_{5/6} = 1 \pm i \rightarrow e^t\cos t, e^t \sin t, te^t\cos t, te^t \sin t$$

(7) Решити Кошијев проблем: $x''' + x'' = 0$ (3. реда)

$$x(0) = 1, \quad x'(0) = 0, \quad x''(0) = 1$$

$$\lambda^3 + \lambda^2 = 0$$

$$\lambda^2(\lambda + 1) = 0$$

$$\text{OP}: \ x(t) = c_1 + c_2 t + c_3 e^{-t}, \quad c_i \in \mathbb{R}$$

$$\left.\begin{aligned} x(0) &= c_1 + c_3 = 1 \\ x'(0) &= c_2 - c_3 = 0 \\ x''(0) &= c_3 = 1 \end{aligned}\right\} \quad \begin{aligned} c_1 &= 0 \\ c_2 &= c_3 = 1 \end{aligned}$$

$$x_k(t) = t + e^{-t}$$