(1) $X' = AX$

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 2$

$k = 4$ – алгебарска вишеструкост

$(A - 2E)\gamma = 0$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 0 = 0 \\ c = 0 \\ 0 = 0 \\ a = 0 \end{matrix} \Rightarrow \gamma = \begin{bmatrix} 0 \\ b \\ 0 \\ d \end{bmatrix} = b \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\dim(\ker(A - 2E)) = 2 \Rightarrow m = 2$ – геометријска вишеструкост $(m \leq k)$

→ имамо (2) Жорданова блока

пример:
$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda$

$$D = \begin{bmatrix} \boxed{\lambda} & & & \\ & \boxed{\lambda} & & \\ & & \boxed{\lambda} & \\ & & & \boxed{\lambda} \end{bmatrix}, \begin{bmatrix} \lambda & 1 & & \\ & \lambda & & \\ & & \boxed{\lambda} & \\ & & & \boxed{\lambda} \end{bmatrix}, \begin{bmatrix} \lambda & 1 & & \\ & \lambda & & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}, \begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & \\ & & & \boxed{\lambda} \end{bmatrix}, \begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}$$

4   3   2   2   1

$$\Rightarrow D = \begin{bmatrix} 2 & 1 & & \\ & 2 & & \\ & & 2 & 1 \\ & & & 2 \end{bmatrix} \vee D = \begin{bmatrix} 2 & 1 & & \\ & 2 & 1 & \\ & & 2 & \\ & & & 2 \end{bmatrix}$$

2 2   2 3

минимални полином? $\mu(\lambda)$

$\varphi(\lambda) = \det(A - \lambda E) = (\lambda - 2)^4$

$\mu \mid \varphi \Rightarrow \mu(\lambda) = (\lambda - 2)^l, \; l \in \{1, 2, 3, 4\}$

$\mu(A)$:

$l = 1$: $(A - 2E)^1 \neq 0$

$l = 2$: $(A - 2E)^2 = 0 \Rightarrow l = 2 \Rightarrow \mu(\lambda) = (\lambda - 2)^2 \Rightarrow \deg \mu = 2$

$\Downarrow$

(2) је ред највећег Ж. блока

$$\Rightarrow D = \begin{bmatrix} 2 & 1 & & \\ & 2 & & \\ & & 2 & 1 \\ & & & 2 \end{bmatrix}$$

γ_1 γ_3? γ_2 γ_4?

$\parallel \gamma_1$ $\parallel \gamma_2$

[0]   [0]   [0]

$$v = \begin{bmatrix}0\\b\\0\\d\end{bmatrix} = b\cdot\overset{= v_1}{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}} + d\cdot\overset{= v_2}{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}}$$

v_1 v_3? v_2 v_4?

Уплътнени собс. век. (Генерализирани)

$v_1 \rightsquigarrow v_3$: $(A-2E)v_3 = v_1$

$$\begin{aligned} 0 &= 0 \\ c &= 1 \\ 0 &= 0 \\ a &= 0 \end{aligned} \Rightarrow v_3 = \begin{bmatrix}0\\b\\1\\d\end{bmatrix} \xrightarrow{b=d=0} v_3 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$

$v_2 \rightsquigarrow v_4$: $(A-2E)v_4 = v_2$

$$\begin{aligned} 0 &= 0 \\ c &= 0 \\ 0 &= 0 \\ a &= 1 \end{aligned} \Rightarrow v_4 = \begin{bmatrix}1\\b\\0\\d\end{bmatrix} \xrightarrow{b=d=0} v_4 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

```
>> A=[2 0 0 0; 0 2 1 0; 0 0 2 0; 1 0 0 2

A =

     2     0     0     0
     0     2     1     0
     0     0     2     0
     1     0     0     2

>> [P D]=eig(A)

P =

         0    0.0000         0         0
         0         0    1.0000   -1.0000
         0         0         0    0.0000
    1.0000   -1.0000         0         0


D =

     2     0     0     0
     0     2     0     0
     0     0     2     0
     0     0     0     2
```

$$P = \begin{bmatrix} v_1 & v_3 & v_2 & v_4 \end{bmatrix} = \begin{bmatrix} 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0 \end{bmatrix}. \quad (v_1 v_3 v_2 v_4 \text{ или } v_2 v_4 v_1 v_3)$$

```
>> jordan(A)

ans =

     2     1     0     0
     0     2     0     0
     0     0     2     1
     0     0     0     2

>> [P D]=jordan(A)

P =

     0     1     0     0
     0     0     1     0
     0     0     0     1
     1     0     0     0


D =

     2     1     0     0
     0     2     0     0
     0     0     2     1
     0     0     0     2
```

$e^{tD} = ?$ $\quad e^{tD} = \begin{bmatrix} e^{t\begin{bmatrix}2&1\\&2\end{bmatrix}} & \\ & e^{t\begin{bmatrix}2&1\\&2\end{bmatrix}} \end{bmatrix} = e^{2t}\begin{bmatrix} 1&t&&\\ &1&&\\ &&1&t\\ &&&1 \end{bmatrix}.$

$$e^{t\begin{bmatrix}2&1\\&2\end{bmatrix}} = e^{t\overset{= 2E}{\begin{bmatrix}2&\\&2\end{bmatrix}} + t\cdot\overset{= N}{\begin{bmatrix}&1\\&\end{bmatrix}}} \overset{(2)}{=} e^{t\begin{bmatrix}2&\\&2\end{bmatrix}}\cdot e^{t\begin{bmatrix}&1\\&\end{bmatrix}} = \begin{bmatrix}e^{2t}&\\&e^{2t}\end{bmatrix}\cdot\begin{bmatrix}1&t\\&1\end{bmatrix} = e^{2t}\cdot\begin{bmatrix}1&t\\&1\end{bmatrix}$$

$2E\cdot N = N\cdot 2E$

N-нилпотентна $(\exists k, N^k = 0)$

$N^2 = \begin{bmatrix}&1\\&\end{bmatrix}\begin{bmatrix}&1\\&\end{bmatrix} = 0$

$N^k = 0$, $k \geq 2$

$$\Rightarrow e^{tN} = \underset{k=0}{E} + \underset{k=1}{t\cdot N} + \underbrace{0}_{k\geq 2} = \begin{bmatrix}1&t\\&1\end{bmatrix}$$

ОР: $X(t) = P\cdot e^{tD}\cdot c$, $c \in \mathbb{R}^4$

② $X' = AX$

$$A = \begin{bmatrix} 2&0&0&0\\ 0&2&0&0\\ 0&0&2&1\\ 1&0&0&2 \end{bmatrix} \qquad \lambda_1=\lambda_2=\lambda_3=\lambda_4=2, \quad k=4$$

$$(A-2E)v=0 \rightsquigarrow v \in \mathrm{Lin}\left\{ \overset{=v_1}{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}}, \overset{=v_2}{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}} \right\} \Rightarrow m=2 \Rightarrow 2 \text{ Ж. блока}$$

$$\begin{bmatrix}2&1&&\\&2&&\\&&2&1\\&&&2\end{bmatrix} \vee \begin{bmatrix}2&1&&\\&2&1&\\&&2&\\&&&2\end{bmatrix}$$

$\varphi(\lambda) = (\lambda-2)^4$

$\mu(\lambda) = (\lambda-2)^l$

$\mu(A)=0:$ $(A-2E)^1 \neq 0$, $(A-2E)^2 \neq 0$, $(A-2E)^3 = 0$ $\Rightarrow \deg\mu = 3 \Rightarrow$ ред највећег Ж. блока је $3 \Rightarrow D = \begin{bmatrix}2&1&&\\&2&1&\\&&2&\\&&&\boxed{2}\end{bmatrix}$.

v_1 v_2

уопштени за v_2: $(A-2E)v = v_2 \Rightarrow \begin{matrix}0=0\\0=1\\d=0\\a=0\end{matrix}$ ↯ $\Rightarrow$ нема реш. $\Rightarrow \boxed{v_2}$ нема уопштени $\Rightarrow v_2$ одг. блоку реда 1

$v_1 \quad v_2$
$\rightsquigarrow$
v_3
$\rightsquigarrow$
v_4

v_1: $(A-2E)v_3 = v_1$

$$\begin{matrix}0=0\\0=0\\d=1\\a=0\end{matrix} \Rightarrow v_3 = \begin{bmatrix}0\\b\\c\\1\end{bmatrix} \xrightarrow{b=c=0} v_3 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}.$$

v_3: $(A-2E)v_4 = v_3$

$$\begin{matrix}0=0\\0=0\\d=0\\a=1\end{matrix} \Rightarrow v_4 = \begin{bmatrix}1\\b\\c\\0\end{bmatrix} \xrightarrow{b=c=0} v_4 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}.$$

```
>> A=[2 0 0 0; 0 2 0 0; 0 0 2 1; 1 0 0 2]

A =

     2     0     0     0
     0     2     0     0
     0     0     2     1
     1     0     0     2

>> [P D]=jordan(A)

P =

     0     0     1     0
     0     0     0     1
     1     0     0     0
     0     1     0     0


D =

     2     1     0     0
     0     2     1     0
     0     0     2     0
     0     0     0     2
```

$$P = \begin{bmatrix} v_1 & v_3 & v_4 & v_2 \end{bmatrix} = \begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix}$$

$$D = \begin{bmatrix}B&\\&2\end{bmatrix} \Rightarrow e^{tD} = \begin{bmatrix}e^{tB}&\\&e^{2t}\end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{bmatrix}$$

$$e^{tB} = e^{t\begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix} + t\cdot \overset{N}{\begin{bmatrix} & 1 & \\ & & 1 \\ & & \end{bmatrix}}} \overset{(2)}{=} e^{t\begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix}} \cdot e^{tN} = e^{2t}\cdot E \cdot \left(E + tN + \frac{t^2}{2}N^2\right) = e^{2t}\cdot \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ & 1 & t \\ & & 1 \end{bmatrix}.$$

$2E\cdot N = N\cdot 2E$

$$N^2 = \begin{bmatrix} & & 1 \\ & & \\ & & \end{bmatrix}$$

$$N^3 = O, \quad N^k = O, \; k \geq 3$$

$$e^{tD} = e^{2t}\cdot \begin{bmatrix} 1 & t & \frac{t^2}{2} & \\ & 1 & t & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \quad \text{ОР}: X(t) = P\cdot e^{tD}\cdot c, \; c \in \mathbb{R}^4$$

③ $X' = AX$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix} \qquad \lambda_1 = \lambda_2 = \lambda_3 = 2, \quad k = 3$$

$$(A-2E)\gamma = 0 \Rightarrow \gamma = a\cdot \underset{\sim}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} + b\cdot \underset{\sim}{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}} \Rightarrow m = \dim\ker(A-2E) = 2 \Rightarrow 2 \text{ блока}$$

$k=3, m=2$: једино $D = \begin{bmatrix} 2 & 1 & \\ & 2 & \\ & & 2 \end{bmatrix}$ (за вежбу: $\deg \mu = 2$)

$$e^{tD} = \begin{bmatrix} e^{t\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix}} & \\ & e^{2t} \end{bmatrix} = e^{2t}\cdot \begin{bmatrix} 1 & t & \\ & 1 & \\ & & 1 \end{bmatrix}.$$

$P = ?$ $\gamma_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\gamma_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ ⤳ треба 1 уопштени ⤳ да ли одг. γ_1 или γ_2?

$(A-2E)\gamma_3 = \gamma_1$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} b+c &= 1 \\ -b-c &= 0 \\ b+c &= 0 \end{aligned} \quad ↯$$

$\Rightarrow \gamma_1$ нема уопш.

$(A-2E)\gamma_3 = \gamma_2$

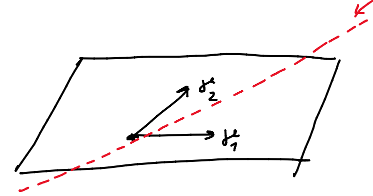
$$\begin{aligned} b+c &= 0 \\ -b-c &= 1 \\ b+c &= -1 \end{aligned} \quad ↯$$

$\Rightarrow \gamma_2$ нема уопш.

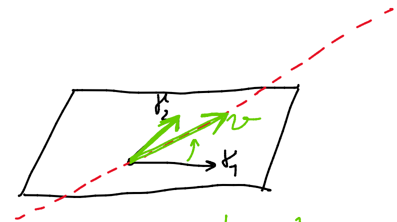
сопствени потпростор:

$= \mathcal{L}in\{\gamma_1, \gamma_2\} = Ker(A-2E)$

γ_3 одговара ван потпростору dim=1



мењамо базу од $ker(A-2E)$:



$\{v, \gamma_2\}$ – нова база

$v = ?$ $\quad v \in \mathcal{L}in\{\gamma_1, \gamma_2\} \Rightarrow v = \alpha \cdot \gamma_1 + \beta \cdot \gamma_2$

→ v има свој уопштени

$\exists \gamma_3 : \quad (A-2E)\gamma_3 = v$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\left.\begin{matrix} b + c = \alpha \\ -b - c = \beta \end{matrix}\right\} \Rightarrow \alpha = -\beta$$

$$b + c = -\beta$$

нпр. $\alpha = 1, \beta = -1$

$$\left.\begin{matrix} b + c = 1, \; b = 1, c = 0, \; a = 0 \end{matrix}\right\} \Rightarrow v = \gamma_1 - \gamma_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \gamma_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

сопствени: v и γ_2

уопштени: γ_3

$$D = \begin{bmatrix} \boxed{\begin{matrix} 2 & 1 \\ & 2 \end{matrix}} & \\ & \boxed{2} \end{bmatrix}$$

$$P = \begin{bmatrix} \boxed{v \; \gamma_3} & \boxed{\gamma_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

```
>> A=[2 1 1; 0 1 -1; 0 1 3]

A =

     2     1     1
     0     1    -1
     0     1     3

>> [P D]=jordan(A)

P =

     1     0     0
    -1     1    -1
     1     0     1


D =

     2     1     0
     0     2     0
     0     0     2
```

ОР: $X(t) = P \cdot e^{tD} \cdot c, \; c \in \mathbb{R}^3$

(4) $X' = AX$

$$A = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \qquad \lambda_1 = \lambda_2 = 2$$

$$\lambda_{3/4} = 1 \pm i$$

$\underline{\lambda_1 = \lambda_2 = 2}$ $(k=2)$

$$(A-2E)v_1 = 0 \Rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow m = 1 \Rightarrow \begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix}$$

$$(A-2E)v_2 = v_1 \Rightarrow v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$\underline{\lambda_{3/4} = 1 \pm i}$

$1+i$

$\alpha + i\beta$ $\rightsquigarrow \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$(\alpha = 1, \beta = 1)$

$$(A-(1+i)E)v_3 = 0 \Rightarrow v_3 = \begin{bmatrix} 1 \\ -2 \\ -1+i \\ -2i \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 0 \end{bmatrix} + i \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 1 & & \\ & 2 & & \\ & & 1 & 1 \\ & & -1 & 1 \end{bmatrix} \rightsquigarrow e^{tD} = \begin{bmatrix} e^{t\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix}} & \\ & e^{t\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}} \end{bmatrix} = \begin{bmatrix} e^{2t} \cdot \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix} & \\ & e^{t} \cdot R_t \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 & 0 \\ 0 & 0 & e^{t}\cos t & e^{t}\sin t \\ 0 & 0 & -e^{t}\sin t & e^{t}\cos t \end{bmatrix}.$$

$$P = \begin{bmatrix} v_1 & v_2 & \mathrm{Re}\, v_3 & \mathrm{Im}\, v_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

OR: $X(t) = P \cdot e^{tD} \cdot c, \; c \in \mathbb{R}^4$