① $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Наћи $\det(e^{e^A})$.

$$A, e^A, e^{e^A} \in M_3(\mathbb{R})$$

$$(6) \Rightarrow \det(e^{e^A}) = e^{tr(e^A)}$$

$$e^A = ?$$

$$A = \underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}}_{E} + \underbrace{\begin{bmatrix} & & 1 \\ & & \\ 1 & & \end{bmatrix}}_{B}$$

$$EB = B = BE$$

$$\Downarrow (2)$$

$$e^A = e^{E+B} = e^E \cdot e^B$$

$$E = diag\{1,1,1\} \Rightarrow e^E = diag\{e,e,e\} = e \cdot E$$

$$B^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = B$$

$$B^4 = B^2$$
$$\vdots$$

индукцијом: $B^0 = E$

$B^{2l} = B^2, \; l \geq 1$

$B^{2l+1} = B, \; l \geq 0$

$$e^B = \sum_{k=0}^{\infty} \frac{B^k}{k!} = \underset{k=0}{E} + \sum_{l=1}^{\infty} \frac{B^{2l}}{(2l)!} + \sum_{l=0}^{\infty} \frac{B^{2l+1}}{(2l+1)!} = E + B^2 \cdot \Big(\overbrace{\sum_{l=0}^{\infty} \frac{1}{(2l)!}}^{ch 1} - 1\Big) + B \cdot \overbrace{\sum_{l=0}^{\infty} \frac{1}{(2l+1)!}}^{sh 1} = \begin{bmatrix} ch 1 & 0 & sh 1 \\ 0 & 1 & 0 \\ sh 1 & 0 & ch 1 \end{bmatrix}$$

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = \Pi + H$$

$$e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = \Pi - H$$

$$\Pi = \frac{e+e^{-1}}{2} = ch 1$$

$$H = \frac{e-e^{-1}}{2} = sh 1$$

$$e^A = eE \cdot e^B = e \cdot e^B \Rightarrow tr(e^A) = e \cdot tr(e^B) = e \cdot (ch 1 + 1 + ch 1) = e(e + e^{-1} + 1) = e^2 + e + 1$$

$$\det(e^{e^A}) = e^{tr(e^A)} = e^{e^2+e+1}.$$

② Нека је $A=[a_{ij}]_{i,j=1}^{n}\in M_n(\mathbb{R})$, $a_{ij}\geq 0, \forall i\neq j$. Нека је $B=e^{A}=[b_{ij}]_{i,j=1}^{n}\in M_n(\mathbb{R})$.

Доказати $b_{ij}\geq 0, \forall i,j$.

$$A=\begin{bmatrix} ? & \geq 0 \\ \geq 0 & ? \end{bmatrix} \rightsquigarrow B=e^{A}=\begin{bmatrix} \geq 0 \end{bmatrix}.$$

$$\circledast\quad \forall i,j,\ a_{ij}\geq 0 \Rightarrow b_{ij}\geq 0, \forall i,j$$

нпр. $A=\begin{bmatrix} -2 & & \\ & -8 & \\ & & 9 \end{bmatrix} = \overset{X}{\underset{\sim E}{\begin{bmatrix} -8 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -8 \end{bmatrix}}} + \overset{Y}{\begin{bmatrix} 6 & & \\ & 0 & \\ & & 17 \end{bmatrix}}$

$$XY = ME\cdot Y = MYE = Y\cdot ME = YX$$
$$\Downarrow (2)$$
$$e^{A}=e^{X}\cdot e^{Y}$$

$A=X+Y$

$X=M\cdot E$

$M=\min\limits_{1\leq i\leq n}\{a_{ii}\}$

$Y=A-X=A-M\cdot E=[y_{ij}]_{i,j=1}^{n}$

$$y_{ij}=\begin{cases} \underset{\geq 0}{a_{ij}}, & i\neq j \\ \underset{\geq 0}{a_{ii}-M}, & i=j \end{cases} \Rightarrow Y=[\geq 0] \overset{\circledast}{\Rightarrow} e^{Y}=[\geq 0]$$

$$X=\operatorname{diag}\{M,\dots,M\} \Rightarrow e^{X}=\operatorname{diag}\{e^{M},\dots,e^{M}\}=e^{M}\cdot E$$

$$B=e^{A}=e^{M}\cdot E\cdot e^{Y}=\underbrace{e^{M}}_{\geq 0}\cdot \overbrace{e^{Y}}^{=[\geq 0]}=[\geq 0] \Rightarrow b_{ij}\geq 0\ \forall i,j$$

Решавање ИСДЈКК свођењем на Жорданову форму

$X'=AX$, $A\in M_n(\mathbb{R})$

ОР: $X(t)=e^{tA}\cdot C$, $C\in\mathbb{R}^{n}$

матрица прелаза $\quad\quad$ $\exists P^{-1}$

$A\sim D$, $A=P\cdot D\cdot P^{-1}$, $P\in GL_n(\mathbb{R})$

у Жордановој нормалној форми

$$e^{tA}=e^{tPDP^{-1}}\overset{(7)}{=}Pe^{tD}P^{-1}$$

- у жордановој нормалној форми

$$e^{tA} = e^{tPDP^{-1}} \overset{(7)}{=} P\underbrace{e^{tD}}_{\text{знамо да израчунамо}}P^{-1}.$$

блокови

$$D = \begin{bmatrix} \boxed{B_1} & & & \\ & \boxed{B_2} & & \\ & & \ddots & \\ & & & \boxed{B_k} \end{bmatrix} \to \text{блок-дијагонална матрица}$$

$\lambda \in \mathbb{R}$ сопс. вр. од A: $\begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}$ $\quad [\lambda] \to$ дијагонална

$\alpha + i\beta \in \mathbb{C}$ сопс. вр. од A: $\begin{bmatrix} \boxed{R} & \boxed{E} & & \\ & \boxed{R} & \ddots & \\ & & \ddots & \boxed{E} \\ & & & \boxed{R} \end{bmatrix}$, где су $R = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}_{2\times 2}$

$(\beta \neq 0)$

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$$

3)
$$x_1' = x_1 - x_2 + x_3$$
$$x_2' = x_1 + x_2 - x_3$$
$$x_3' = 2x_1 - x_2$$

а) Наћи ОР.

б) Наћи ПР $X(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

а) $X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$

$X' = AX$, $\quad A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

$$\begin{vmatrix} 1-\lambda & -1 \\ -1 & -\lambda \end{vmatrix}$$

$$\det(A - \lambda E) = 0 \Rightarrow 0 = \begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & 1-\lambda & -1 \\ 2 & -1 & -\lambda \end{vmatrix} = (1-\lambda)\left((1-\lambda)(-\lambda) - (-1)^2\right) - 1(\lambda + 1) + 2\left(\overbrace{1 - (1-\lambda)}^{\lambda}\right) =$$

$$= (1-\lambda)(\lambda^2 - \lambda - 1)\underbrace{-\lambda - 1 + 2\lambda}_{\lambda - 1} =$$

$$= (1-\lambda)(\lambda^2 - \lambda - 1 - 1) =$$

$$= (1-\lambda)(\lambda - 2)(\lambda + 1)$$

```
>> A=[1 -1 1; 1 1 -1; 2 -1 0]

A =

     1    -1     1
     1     1    -1
     2    -1     0

>> eig(A)

ans =

   -1.0000
    1.0000
    2.0000
```

$\lambda_1 = -1$, $\lambda_2 = 1$, $\lambda_3 = 2$

$$\Rightarrow D = \begin{bmatrix} -1 & & \\ & 1 & \\ & & 2 \end{bmatrix}$$

```
>> [P D] = eig(A)

P =

    0.1690   -0.5774    0.7071
   -0.5071   -0.5774    0.0000
   -0.8452   -0.5774    0.7071

D =

   -1.0000         0         0
         0    1.0000         0
         0         0    2.0000
```

$\gamma_1 \quad \gamma_2 \quad \gamma_3$

$\lambda_1 \quad \lambda_2$

$P = ? \quad P = \begin{bmatrix} \gamma_1\downarrow & \gamma_2\downarrow & \gamma_3\downarrow \end{bmatrix}$

$$(A - \lambda_1 E)\gamma_1 = \vec{0}$$

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} 2a - b + c &= 0 \\ a + 2b - c &= 0 \\ 2a - b + c &= 0 \end{aligned}$$

$\vdots$

$$b = -3a \qquad c = -5a$$

$$\begin{bmatrix} a \\ -3a \\ -5a \end{bmatrix} = a \cdot \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix} \underset{(a=1)}{=} \gamma_1$$

$$\lambda_2 \rightsquigarrow \gamma_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \lambda_3 \rightsquigarrow \gamma_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 1 & 0 \\ -5 & 1 & 1 \end{bmatrix}$$

```
>> P = [1 1 1; -3 1 0; -5 1 1]

P =

     1     1     1
    -3     1     0
    -5     1     1

>> inv(P)

ans =

    0.1667    0.0000   -0.1667
    0.5000    1.0000   -0.5000
    0.3333   -1.0000    0.6667
```

```
>> det(P)

ans =

     6
```

$P^{-1} = ?$

$$P^{-1} = \frac{1}{\underset{=6}{\det P}} \cdot \operatorname{Adj} P = \frac{1}{6} \cdot \begin{bmatrix} +\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} -3 & 0 \\ -5 & 1 \end{vmatrix} & +\begin{vmatrix} -3 & 1 \\ -5 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ -5 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ -5 & 1 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ -3 & 0 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ -3 & 1 \end{vmatrix} \end{bmatrix}^T = \frac{1}{6} \cdot \begin{bmatrix} 1 & 3 & 2 \\ 0 & 6 & -6 \\ -1 & -3 & 4 \end{bmatrix}^T = \frac{1}{6} \cdot \begin{bmatrix} 1 & 0 & -1 \\ 3 & 6 & -3 \\ 2 & -6 & 4 \end{bmatrix}$$

$$D \rightsquigarrow e^{tD} = \begin{bmatrix} e^{-t} & & \\ & e^{t} & \\ & & e^{2t} \end{bmatrix}$$

ОР: $X(t) = P e^{tD} \underbrace{P^{-1} \cdot c}_{c_1} = P e^{tD} c_1$

$c_1, c \in \mathbb{R}^3$

б) $X(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$X(t) = P e^{tD} c_1$

$$X(0) = P\cdot E\cdot c_1 = Pc_1 = \begin{bmatrix}1\\1\\1\end{bmatrix} \Rightarrow c_1 = P^{-1}\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

$$X(t) = Pe^{tD}P^{-1}\cdot c$$

$$X(0) = \underbrace{P\cdot E\cdot P^{-1}}_{E}\cdot c = c = \begin{bmatrix}1\\1\\1\end{bmatrix} \rightsquigarrow \text{np: } X(t) = Pe^{tD}P^{-1}\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

④ $X' = AX,\quad A = \begin{bmatrix}-3 & 0 & 0\\ 0 & 3 & -2\\ 0 & 1 & 1\end{bmatrix}$

```
>> A=[-3 0 0; 0 3 -2; 0 1 1]

A =

    -3     0     0
     0     3    -2
     0     1     1

>> eig(A)

ans =

   2.0000 + 1.0000i
   2.0000 - 1.0000i
  -3.0000 + 0.0000i
```

$\lambda_{1/2} = 2 \pm i$

$\lambda_3 = -3$

$\lambda_1 = 2+i \longrightarrow \begin{bmatrix}2 & 1\\ -1 & 2\end{bmatrix}$

$$D = \begin{bmatrix}\boxed{\begin{matrix}2 & 1\\ -1 & 2\end{matrix}}_{2\times 2} & \\ & \boxed{-3}_{1\times 1}\end{bmatrix}$$

λ_1 $\quad$ λ_3

$, \quad P = ? \quad \lambda_3 = -3 \rightsquigarrow v_3 = \begin{bmatrix}1\\0\\0\end{bmatrix}$

$\lambda_1 = 2+i \rightsquigarrow v_1 \in \mathbb{C}^3 \quad \left((A - \lambda_1 E)v_1 = \vec{o}\right)$

$\Downarrow$

$\operatorname{Re} v_1,\ \operatorname{Im} v_1$

$$(A - (2+i)E)v_1 = \vec{o}$$

$$\begin{bmatrix}-3-(2+i) & 0 & 0\\ 0 & 3-(2+i) & -2\\ 0 & 1 & 1-(2+i)\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}, \quad a, b, c \in \mathbb{C}$$

$(-5-i)a = 0 \Rightarrow a = 0$

$(1-i)b - 2c = 0$

$\cancel{b + (-1-i)c = 0}\ /\cdot(1-i) \longrightarrow (1-i)b + \underbrace{(1-i)(-1-i)}_{-2}c = 0$

$(1-i)b = 2c$

$c = 1-i \quad , b = 2 \qquad \Rightarrow v_1 = \begin{bmatrix}0\\2\\1-i\end{bmatrix} = \begin{bmatrix}0\\2\\1\end{bmatrix} + \begin{bmatrix}0\\0\\-1\end{bmatrix}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \operatorname{Re} v_1 \qquad \operatorname{Im} v_1$

$$P = \begin{bmatrix}\operatorname{Re} v_1\downarrow & \operatorname{Im} v_1\downarrow & v_3\downarrow\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\ 2 & 0 & 0\\ 1 & -1 & 0\end{bmatrix}$$

```
>> D=[2 1 0; -1 2 0; 0 0 -3]

D =
```

```
>> [P D] = eig(A)

P =

   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i
   0.8165 + 0.0000i   0.8165 + 0.0000i   0.0000 + 0.0000i
   0.4082 - 0.4082i   0.4082 + 0.4082i   0.0000 + 0.0000i

D =
```

$v_1 \qquad v_2 \qquad v_3$

```
>> D=[2 1 0; -1 2 0; 0 0 -3]

D =

     2     1     0
    -1     2     0
     0     0    -3

>> P=[0 0 1; 2 0 0; 1 -1 0]

P =

     0     0     1
     2     0     0
     1    -1     0

>> P*D*inv(P)

ans =

    -3     0     0
     0     3    -2
     0     1     1
```

Провера:

$A = PDP^{-1}$

```
   0.8165 + 0.0000i   0.8165 + 0.0000i   0.0000 + 0.0000i
   0.4082 - 0.4082i   0.4082 + 0.4082i   0.0000 + 0.0000i
```

$v_1 \qquad v_2 \qquad v_3$

```
D =
```

λ_1

```
   2.0000 + 1.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   2.0000 - 1.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i  -3.0000 + 0.0000i
```

$\lambda_2 \qquad \lambda_3$

$$\lambda_2 = \overline{\lambda_1} \Rightarrow v_2 = \overline{v_1}$$

$$e^{tD} = ?$$

$$D = \begin{bmatrix} B_1 & \\ & B_2 \end{bmatrix} \rightsquigarrow D^k = \begin{bmatrix} B_1^k & \\ & B_2^k \end{bmatrix} \rightsquigarrow e^{tD} = \begin{bmatrix} e^{tB_1} & \\ & e^{tB_2} \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} -3 \end{bmatrix}$$

$$e^{tD} = \begin{bmatrix} e^{t\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}} & \\ & e^{t[-3]} \end{bmatrix} = \begin{bmatrix} e^{2t} \cdot \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} & \\ & e^{-3t} \end{bmatrix} = \begin{bmatrix} e^{2t}\cos t & e^{2t}\sin t & 0 \\ -e^{2t}\sin t & e^{2t}\cos t & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix}.$$

подсетник: $e^{t\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}} = e^{t\alpha} \cdot R_{t\beta}$

ОР: $X(t) = P \cdot e^{tD} \cdot c \,,\; c \in \mathbb{R}^3$