(*) На предавањима је Пикар форм. за C^1 ф-је $\rightsquigarrow$ у доказу се користи лок. лип.

→ предавања: $F(x,t) \rightsquigarrow F(x)$, $\|F(x)-F(y)\| \le L\cdot\|x-y\|$

→ вежбе: $F(x,t)$, $\|F(x,t)-F(y,t)\| \le L\cdot\|x-y\|$

(1) $x' = x(1-x)$. Без решавања:

а) Свако решење т.д. $x(0)=a\in(0,1) \Rightarrow (\forall t)\ 0<x(t)<1$.

б) Наћи $\lim_{t\to\infty} x(t)$ у зависности од $x(0)=a\in\mathbb{R}$.

а) $\left.\begin{array}{l} x' = x(1-x) \\ x(0)=a \end{array}\right\}$ Пикар

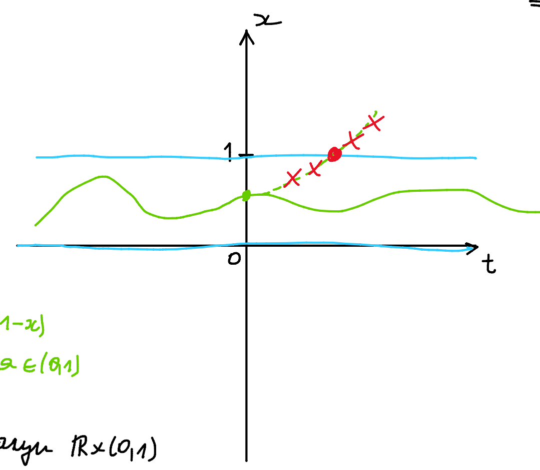
$F(x,t) = x(1-x) \longrightarrow C^1$ ф-ја!

↳ аутономно (нема t)

$F'_x = (x(1-x))' = 1-2x \in C(\mathbb{R})$

$\Rightarrow F$ је $C^1 \Rightarrow$ важи Пикар $\Rightarrow$ једин. реш.

$\Rightarrow$ реш. се не секу



$x\equiv 0$ ✓ $\quad 0' = 0\cdot 1$

$x \equiv 1$ ✓ $\quad 1' = 1\cdot 0$

$x' = x(1-x)$

$x(0) = a\in(0,1)$

$\Rightarrow x(t)$ је у траци $\mathbb{R}\times(0,1)$

$\Rightarrow 0 < x(t) < 1$.

б) $x(0)=a$

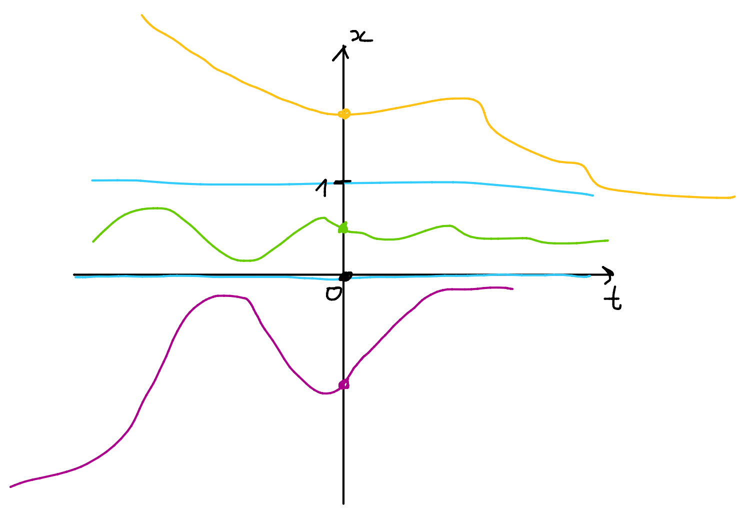
1° $a\in(0,1) \Rightarrow x(t)\in(0,1)$

2° $a>1 \Rightarrow x(t)>1$

3° $a<0 \Rightarrow x(t)<0$

4° $a=0 \Rightarrow x(t)=0$

5° $a=1 \Rightarrow x(t)=1$



1° $x(t)\in(0,1)$

$\underset{\in(0,1)}{x}' = x\underbrace{(1-x)}_{\in(0,1)} \Rightarrow x' \in (0,1) \Rightarrow x' > 0 \Rightarrow x \nearrow$

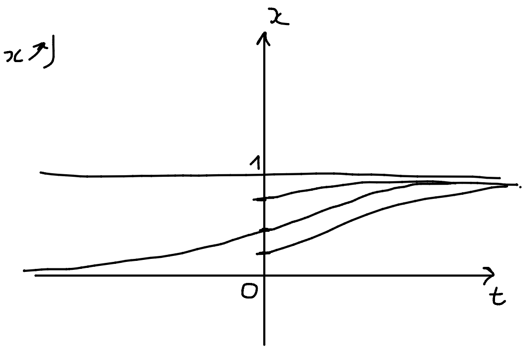
хор. асимптота мора да постоји (огр, C^1, $x \nearrow$)

$\Rightarrow x' \to 0$

$x \to 0 \lor x \to 1$

због $x \nearrow$

$\Rightarrow$ сва решења $\lim\limits_{t\to\infty} x(t) = 1$.



2° $x(t) > 1$

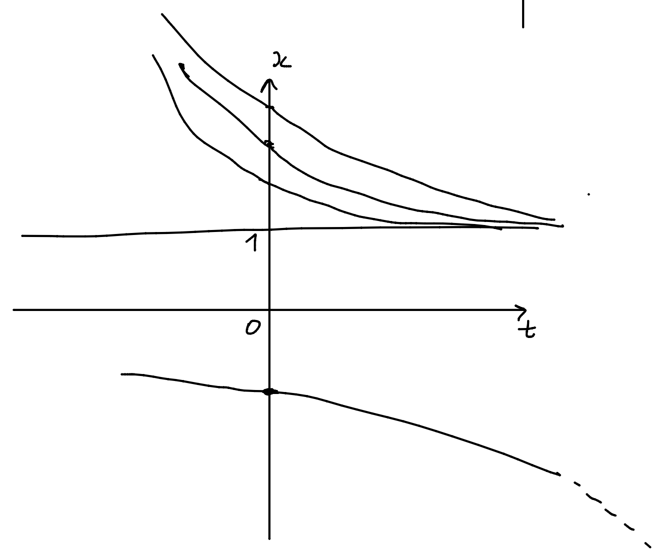
$x' = \underbrace{x}_{>0}\underbrace{(1-x)}_{<0} \Rightarrow x' < 0 \Rightarrow x \searrow$

хор. асимпт?

$x' \to 0 \Rightarrow x \to 1$

$x > 1$

$\lim\limits_{t\to\infty} x(t) = 1$



3° $x(t) < 0$

$x' = \underbrace{x}_{<0}\underbrace{(1-x)}_{>0} \Rightarrow x' < 0 \Rightarrow \underline{x \searrow}$

не може имати хор. ас. десно $\Rightarrow \lim\limits_{t\to\infty} x(t) = -\infty$

$$\lim_{t\to\infty} x(t) = \begin{cases} 1, & a > 0 \\ 0, & a = 0 \\ -\infty, & a < 0 \end{cases}$$

(2) Доказати да $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (\sqrt{x^2+y^2}, \sqrt[4]{x^2+y^2})$ није локално Липшицова ни у једној околини $(0,0)$.

Рј. $X' = f(X)$, $X = (x,y)$

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} \sqrt{x^2+y^2} \\ \sqrt[4]{x^2+y^2} \end{bmatrix}$$

$\Updownarrow$

$x' = \sqrt{x^2+y^2}$

$y' = \sqrt[4]{x^2+y^2}$

не важи Пикар за $X(t_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$\Updownarrow$ не важи Пикар за $X(t_0)=\begin{bmatrix}0\\0\end{bmatrix}$.

$$x' = \sqrt{x^2+y^2}$$
$$y' = \sqrt[4]{x^2+y^2}$$

$$X_1=(x_1,y_1)\ ,\ X_2=(x_2,y_2)$$

$$\|f(X_1)-f(X_2)\| \le L\cdot\|X_1-X_2\|$$

узмемо $X_2=(0,0)$: $f(X_2)=X_2=(0,0)$ $\qquad$ $\|f(X_1)\|\le L\cdot\|X_1\|$, X_1 у околини $(0,0)$

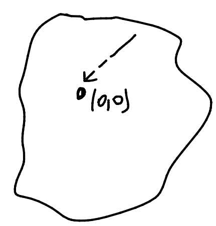
$$\left\|\left(\sqrt{x_1^2+y_1^2}\,,\ \sqrt[4]{x_1^2+y_1^2}\right)\right\| \le L\cdot\|(x_1,y_1)\|$$ $\qquad$ $\|\cdot\| \to$ узимамо еуклидску $\|\cdot\|_2$

$$\sqrt{\left(\sqrt{x_1^2+y_1^2}\right)^2+\left(\sqrt[4]{x_1^2+y_1^2}\right)^2} \le L\cdot\sqrt{x_1^2+y_1^2} \quad /^2$$

$$x_1^2+y_1^2+\sqrt{x_1^2+y_1^2} \le L^2\cdot(x_1^2+y_1^2) \quad /:(x_1^2+y_1^2)\ ,\ X_1\neq(0,0)$$

$$1+\boxed{\frac{1}{\sqrt{x_1^2+y_1^2}}} \le L^2 \qquad \to\infty \qquad ↯$$

$$\lim_{(x_1,y_1)\to(0,0)} \frac{1}{\sqrt{x_1^2+y_1^2}}=\infty$$



(3) Формирати низ итерација из доказа Пикарове Т за проблем: $x'=\frac{x}{t}$, $x(t_0)=x_0$, $t_0>0$.

$$x_0(t)\equiv x_0 \qquad\qquad F(x,t)=\frac{x}{t}$$

$$x_{n+1}(t)=x_0+\int_{t_0}^{t} F(x_n(s),s)\,ds \quad ,\ x_n\to x_\infty$$
↳ решење ду.

$$x_1(t)=x_0+\int_{t_0}^{t}F(x_0(s),s)\,ds = x_0+\int_{t_0}^{t}\frac{x_0}{s}\,ds = x_0+x_0(\ln s)\Big|_{t_0}^{t} = x_0+x_0(\ln t-\ln t_0)=x_0+x_0\ln\frac{t}{t_0}$$

$$x_2(t)=x_0+\int_{t_0}^{t}F(x_1(s),s)\,ds = x_0+\int_{t_0}^{t}\frac{x_0+x_0\ln\frac{s}{t_0}}{s}\,ds = x_0+x_0\ln s\Big|_{t_0}^{t}+x_0\int_{t_0}^{t}\frac{\ln\frac{s}{t_0}}{s}\,ds = x_0+x_0\ln\frac{t}{t_0}+x_0\int_{0}^{\ln\frac{t}{t_0}}u\,du =$$

$$=x_0+x_0\ln\frac{t}{t_0}+x_0\cdot\frac{1}{2}\ln^2\frac{t}{t_0}$$

$$u=\ln\frac{s}{t_0}$$
$$du=\frac{1}{\frac{s}{t_0}}\cdot\frac{1}{t_0}\,ds=\frac{ds}{s}$$

$$du = \frac{1}{\frac{s}{t_0}} \cdot \frac{1}{t_0} ds = \frac{ds}{s}$$

$$x_3(t) = x_0 + \int_{t_0}^{t} F(x_2(s), s) ds = x_0 + \int_{t_0}^{t} \frac{x_0 + x_0 \ln\frac{s}{t_0} + \frac{x_0}{2}\ln^2\frac{s}{t_0}}{s} ds = x_0 + x_0 \ln\frac{t}{t_0} + \frac{x_0}{2}\ln^2\frac{t}{t_0} + \frac{x_0}{2}\cdot\int_{t_0}^{t} \frac{\ln^2\frac{s}{t_0}}{s} ds =$$

$$= x_0 + x_0 \ln\frac{t}{t_0} + \frac{x_0}{2}\ln^2\frac{t}{t_0} + \frac{x_0}{3!}\ln^3\frac{t}{t_0}.$$

$$\int_{t_0}^{t} \frac{\ln^k\frac{s}{t_0}}{s} ds = \int_0^{\ln\frac{t}{t_0}} u^k du = \frac{u^{k+1}}{k+1}\Big|_0^{\ln\frac{t}{t_0}} = \frac{\ln^{k+1}\frac{t}{t_0}}{k+1}$$

$$u = \ln\frac{s}{t_0}, \quad du = \frac{ds}{s} \qquad k \geq 0$$

индукцијом: $x_n(t) = x_0 + x_0 \ln\frac{t}{t_0} + \dots + \frac{x_0}{n!}\ln^n\frac{t}{t_0} = x_0 \cdot \sum_{k=0}^{n} \frac{\ln^k\frac{t}{t_0}}{k!}$.

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} \frac{x_n(s)}{s} ds = x_0 + \int_{t_0}^{t} \frac{x_0 \cdot \sum_{k=0}^{n} \frac{\ln^k\frac{s}{t_0}}{k!}}{s} ds = x_0 + x_0 \cdot \sum_{k=0}^{n} \frac{1}{k!} \int_{t_0}^{t} \frac{\ln^k\frac{s}{t_0}}{s} ds =$$

$$= x_0 + x_0 \sum_{k=0}^{n} \frac{1}{k!} \cdot \frac{\ln^{k+1}\frac{t}{t_0}}{k+1} = x_0 + x_0 \cdot \sum_{k=0}^{n} \frac{\ln^{k+1}\frac{t}{t_0}}{(k+1)!} = x_0 \cdot \sum_{k=0}^{n+1} \frac{\ln^k\frac{t}{t_0}}{k!} \quad \checkmark$$

$$x_n(t) = x_0 \cdot \sum_{k=0}^{n} \frac{\ln^k\frac{t}{t_0}}{k!} \xrightarrow[n\to\infty]{} x_0 \cdot \exp\left(\ln\frac{t}{t_0}\right) = x_0 \cdot \frac{t}{t_0} = \frac{x_0}{t_0}\cdot t. \qquad e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\text{пр}, \quad x(t) = \frac{x_0}{t_0}\cdot t.$$

Експоненти матрице

$X' = AX$ линеаран систем дј. са константним коефицијентима

$X : \mathbb{R} \to \mathbb{R}^n$

$t \mapsto X(t)$

$X'(t) = A \cdot X(t)$, $X', X \in \mathbb{R}^n$

$A \in M_n(\mathbb{R})$, $A = [a_{ij}]_{i,j=1}^{n}$

1 једн. 1. реда у $\mathbb{R}^n$
$\wr$
n једн 1. реда у $\mathbb{R}$

$[\, x_1(t) \,]$ $x_1'(t) = a_{11}x_1(t) + \dots + a_{1n}x_n(t)$

$$X = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \Rightarrow \begin{matrix} x_1'(t) = a_{11} x_1(t) + \dots + a_{1n} x_n(t) \\ \vdots \qquad \ddots \qquad \vdots \\ x_n'(t) = a_{n1} x_1(t) + \dots + a_{nn} x_n(t) \end{matrix}$$

n једн. 1. реда у $\mathbb{R}$

ОР: $X(t) = e^{tA} \cdot c, \; c \in \mathbb{R}^n, \quad c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \; c_1, \dots, c_n \in \mathbb{R}$

нема смисла:

$X(t) = \underline{c \cdot e^{tA}}, \; c \in \mathbb{R}^2$

$2\times1 \to 2\times1 \cdot 2\times2$

$\exp(A) = e^A$

$\exp : M_n(\mathbb{R}) \to M_n(\mathbb{R}), \quad \exp(A) = e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$

1) Решити систем дј. $X' = AX$, одређивањем e^{tA} у облику реда, ако је:

а) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ б) $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ в) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ г) $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \; a, b \in \mathbb{R}$

$X(t) = e^{tA} \cdot c, \; c \in \mathbb{R}^2$

$e^{tA} = ?$

а) $e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$

$A^k = ?$

$A^2 = A \cdot A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$A^3 = A^2 \cdot A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

индукцијом: $A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, \; k \geq 1$

Б: $k = 1, \quad A^1 = A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ ✓

Х: $A^k = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

К: $k+1, \quad A^{k+1} = A^k \cdot A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1+k \\ 0 & 1 \end{bmatrix}$ ✓

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k \cdot \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}}{k!} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{t^k}{k!} & \sum_{k=0}^{\infty} \frac{t^k}{k!} \cdot k \\ 0 & \sum_{k=0}^{\infty} \frac{t^k}{k!} \end{bmatrix} = \begin{bmatrix} e^t & t \cdot e^t \\ 0 & e^t \end{bmatrix}.$$

$$e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!}$$

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \cdot k = \sum_{k=1}^{\infty} \frac{t^k}{(k-1)!} = t \cdot \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} = t \cdot \sum_{k=0}^{\infty} \frac{t^k}{k!} = t \cdot e^t$$

OP: $X(t) = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \cdot c = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 e^t + c_2 t e^t \\ c_2 e^t \end{bmatrix}$

$c \in \mathbb{R}^2$

$x_1(t) = c_1 e^t + c_2 t e^t$

$x_2(t) = c_2 e^t$

б) домаћи