$\boxed{7}$ ДЈ са тоталним диференцијалом

$$M(t,x)\,dt + N(t,x)\,dx = 0$$

$$x' = \frac{f(t,x)}{g(t,x)} \Leftrightarrow \frac{dx}{dt} = \frac{f(t,x)}{g(t,x)} \Big/ \text{„}dt\text{"} \Leftrightarrow$$

$$g(t,x)\,dx = f(t,x)\,dt \Leftrightarrow f(t,x)\,dt - g(t,x)\,dx = 0$$

ДЈ у диференцијалном облику

$\exists F(t,x)$ — претпоставка

$$dF(t,x) = M(t,x)\,dt + N(t,x)\,dx$$

$$\frac{\partial F}{\partial t}(t,x)\,dt + \frac{\partial F}{\partial x}(t,x)\,dx$$

$$\Rightarrow \left.\begin{aligned} M(t,x) &= \frac{\partial F}{\partial t}(t,x) \quad /'_x \\ N(t,x) &= \frac{\partial F}{\partial x}(t,x) \quad /'_t \end{aligned}\right\} \Rightarrow M'_x = \frac{\partial^2 F}{\partial x\,\partial t} = \frac{\partial^2 F}{\partial t\,\partial x} = N'_t$$

$$\text{ТОТ.Д.} \Rightarrow M'_x = N'_t$$

важи и $\boxed{\Leftarrow}$ ако радимо у просто-повезаној области

↳ тамо где посматрамо, где је деф...

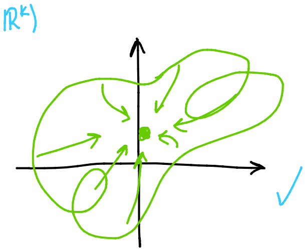
$$0 = M\,dt + N\,dx = dF$$

ОР: $F = c,\ c \in \mathbb{R}$ ← имплицитно задато

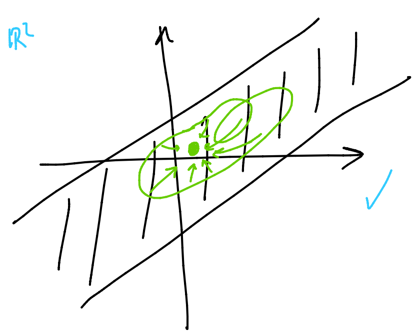
Просто-повезане области (ПП)

$D \subseteq \mathbb{R}^k$ је ПП $\Leftrightarrow$ свака непрекидна затворена крива у D се може непрекидно деформисати у тачку кроз D и D је повезана

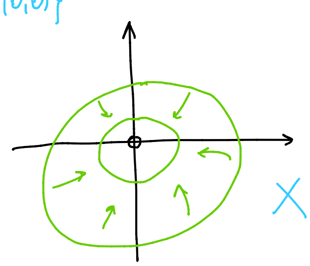
пр. 1) $\mathbb{R}^2$ ($\mathbb{R}^k$)



2) права у $\mathbb{R}^2$



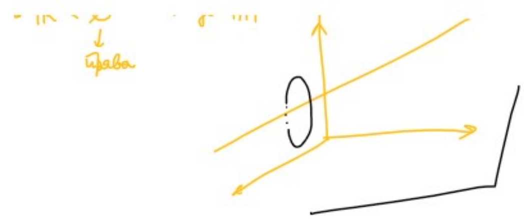
3) $\mathbb{R}^2 \setminus \{(0,0)\}$



„ПП у $\mathbb{R}^2$ ако нема рупа"

не важи у $\mathbb{R}^k$

- $\mathbb{R}^3 \setminus \{(0,0,0)\}$ јесте ПП
- $\mathbb{R}^3 \setminus \ell$ није ПП (ℓ — права)



① а) $2t(1+\sqrt{t^2-x})\,dt - \sqrt{t^2-x}\,dx = 0$

б) $(1+x^2\sin t)\,dt - x\cos^2 t\,dx = 0$

в) $(tx^2+3t^2x)\,dt + (t^3+t^2x)\,dx = 0$

а) $M(t,x) = 2t(1+\sqrt{t^2-x})$

$N(t,x) = -\sqrt{t^2-x}$

$M'_x = 2t\cdot\dfrac{1}{2\sqrt{t^2-x}}\cdot(-1) = \dfrac{-t}{\sqrt{t^2-x}}$

$N'_t = -\dfrac{1}{2\sqrt{t^2-x}}\cdot 2t = \dfrac{-t}{\sqrt{t^2-x}}$

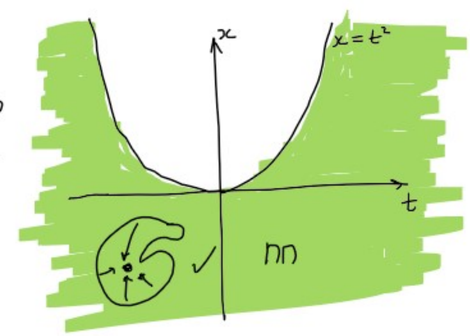
$\Downarrow$

ТОТ.Д.

области?

$t^2-x \geq 0$

$x \leq t^2$



$\Rightarrow \exists F,\ dF = M\,dt + N\,dx\ ,\ F = ?$

$\dfrac{\partial F}{\partial t} = M = 2t + 2t\sqrt{t^2-x}$

$\dfrac{\partial F}{\partial x} = N = -\sqrt{t^2-x}\ \Big/ \int dx$

$F = -\int\sqrt{t^2-x}\,dx = \dfrac{2}{3}(t^2-x)^{3/2} + c(t)$, $c:\mathbb{R}\to\mathbb{R}$

$(c\in\mathbb{R}\ \times)$

$M = \dfrac{\partial F}{\partial t} = \dfrac{2}{3}\cdot\dfrac{3}{2}\cdot(t^2-x)^{1/2}\cdot 2t + c'(t)$

$\| $

$2t + 2t\sqrt{t^2-x}$

$\Rightarrow c'(t) = 2t$ ← ако не добијемо да зависи само од t ⇒ ГРЕШКА (у рачуну)

$\Rightarrow c(t) = t^2$

ОР: $F(t,x) = \dfrac{2}{3}(t^2-x)^{3/2} + t^2 = c,\ c\in\mathbb{R}$

8 Интеграциони фактор

$M(t,x)\,dt + N(t,x)\,dx = 0$

$M'_x \neq N'_t \Rightarrow$ није ТОТ.Д.

Идеја: $\underbrace{\overset{\to\text{ интеграциони фактор}}{\mu(t,x)}\left(M(t,x)\,dt + N(t,x)\,dx\right)}_{\text{јесте тот. д.}} = 0$ $\qquad \mu = ?$

$$\left.\begin{array}{l}\tilde{M} = \mu\cdot M\\ \tilde{N} = \mu\cdot N\end{array}\right\}\quad \tilde{M}'_x = \tilde{N}'_t$$

$\mu(t,x)$ често је у облику $\underset{\mathbb{R}^2\to\mathbb{R}}{\mu} = \mu(w) = \underset{\mathbb{R}\to\mathbb{R}}{\mu}(\underset{\mathbb{R}^2\to\mathbb{R}}{w(t,x)})$

$$\left.\begin{array}{l}\tilde{M}'_x = \frac{\partial}{\partial x}(\mu\cdot M) = \underline{\mu'_x}\cdot M + \mu\cdot M'_x = \mu'(w)\cdot w'_x\cdot M + \mu\cdot M'_x\\ \tilde{N}'_t = \frac{\partial}{\partial t}(\mu N) = \mu'_t\cdot N + \mu\cdot N'_t = \mu'(w)\cdot w'_t\cdot N + \mu\cdot N'_t\end{array}\right\} =$$

$$\Rightarrow \mu'(w)\left(w'_x M - w'_t N\right) = \mu\left(N'_t - M'_x\right)$$

$$\mu'(w) = \frac{d\mu}{dw} \qquad\qquad \frac{d\mu}{\mu} = \boxed{\frac{N'_t - M'_x}{w'_x M - w'_t N}}\,dw$$

↳ мора да зависи само од w

$\}$ РП

$$\int \frac{d\mu}{\mu} = \int \frac{N'_t - M'_x}{w'_x M - w'_t N}\,dw$$

Често:

- $w(t,x) = at + bx$ $\quad (w = t,\ w = x, \ldots)$
- $w(t,x) = a\ln|t| + b\ln|x|$
- $w(t,x) = f(t) + g(x)$
- $w(t,x) = f(t)\cdot g(x)$

$\frac{1}{\mu} = 0$ проверити за изгубљено решење

(2) а) $(g(t) - p(t)x)\,dt - dx = 0$, РЈ познате ← упоредити са ОР линеарне $(w = t)$

б) $2tx\ln x\,dt + (t^2 + x^2\sqrt{x^2+1})\,dx = 0$

в) $x(2 - 3tx^2)\,dt - t(1 + tx^2)\,dx = 0$, на области $G = \{t > 0, x > 0\}$, наћи решење кроз $(2,1)$

г) $(\sqrt{t^2 - x} + 2t)\,dt - dx = 0$, μ у облику $\mu(\underline{t^2 - x})$ $(w = t^2 - x)$

д) $(t+2)\sin x\,dt + 2t\cos x\,dx = 0$ $\qquad (w = x)$

$$\left(x' = \frac{dx}{dt} = -\frac{t+2}{2t}\cdot \operatorname{tg} x \rightsquigarrow \text{РП}\right)$$

б) $M(t,x) = 2tx\ln x$

$N(t,x) = t^2 + x^2\sqrt{x^2+1}$

$$M'_x = 2t\left(x\cdot\frac{1}{x} + 1\cdot\ln x\right) = 2t(1+\ln x) \nparallel$$

$$N'_t = 2t$$

$$\frac{d\mu}{\mu} = \frac{N'_t - M'_x}{w'_x M - w'_t N}\,dw = \frac{-2t\ln x}{w'_x\cdot 2tx\ln x \underbrace{- w'_t\cdot\left(t^2 + x^2\sqrt{x^2+1}\right)}_{=0}}\,dw = -\frac{1}{w'_x\cdot x}\,dw = -\frac{1}{2x^2}\,dw = -\frac{1}{2w}\,dw \quad /\int$$

$w'_t = 0$

$w = w(x)$

$w = x$, $w = \ln x$, $w = x^2$

$$\Rightarrow \ln|\mu| = -\frac{1}{2}\ln|w| \quad (+c)$$

$$\mu = w^{-\frac{1}{2}} = \frac{1}{\sqrt{w}}$$

$(x>0$, због $\ln x)$

$$\mu(t,x) = \mu(w(t,x)) = \frac{1}{\sqrt{w(t,x)}} = \frac{1}{\sqrt{x^2}} = \frac{1}{x}$$

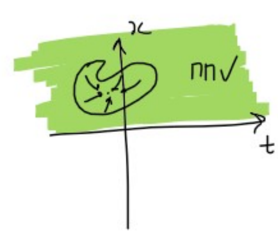
$\left[\frac{1}{\mu} = 0,\ x = 0 \ \lightning,\ x>0\right]$

$$/\cdot\frac{1}{x} \Rightarrow \underbrace{2t\ln x}_{M}\,dt + \underbrace{\left(\frac{t^2}{x} + x\sqrt{x^2+1}\right)}_{N}dx = 0$$

ТОТ. Д.?

$M'_x = N'_t$ ✓

области? $x>0$



$F = ?$ $\left.\begin{array}{l}\frac{\partial F}{\partial t} = M \\ \frac{\partial F}{\partial x} = N\end{array}\right\} \ldots$ ОР: $F(t,x) = t^2\ln x + \frac{1}{3}(x^2+1)^{3/2} = c,\ c\in\mathbb{R}$

в) $\underbrace{x(2-3tx^2)}_{M}dt \underbrace{- t(1+tx^2)}_{N}dx = 0$, на области $G = \{t>0, x>0\}$, наћи решење кроз $(2,1)$

$M'_x = 2 - 9tx^2$

$N'_t = -1 - 2tx^2$ $\#$

$$\frac{d\mu}{\mu} = \frac{(-1 - 2tx^2) - (2 - 9tx^2)}{w'_x \boxed{x}(2 - 3tx^2) + w'_t \cdot \boxed{t}(1 + tx^2)} dw = \frac{-3 + 7tx^2}{b(2 - 3tx^2) + a(1 + tx^2)} dw = \frac{-3 + 7tx^2}{(2b + a) + tx^2(-3b + a)} dw = 1\, dw \quad / \int$$

$= -3 + 7tx^2$

$w = a\ln t + b\ln x$

$w'_x = \frac{b}{x}$

$w'_t = \frac{a}{t}$

$$\left.\begin{aligned} 2b + a &= -3 \\ -3b + a &= 7 \end{aligned}\right\} -$$

$5b = -10 \Rightarrow b = -2$

$a = 1$

$w = \ln t - 2\ln x$

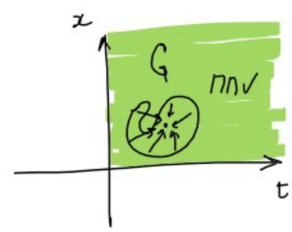
$$\Rightarrow \mu = e^{w} = e^{\ln t - 2\ln x} = \frac{t}{x^2}$$

$/ \cdot \frac{t}{x^2}$

$$\underbrace{\left(2\frac{t}{x} - 3t^2x\right)}_{M} dt - \underbrace{\left(\frac{t^2}{x^2} + t^3\right)}_{N} dx$$

$M'_x = N'_t$

области:



$$\left.\begin{aligned} \frac{\partial F}{\partial t} &= M \\ \frac{\partial F}{\partial x} &= N \end{aligned}\right\} \cdots$$

OP:

$$F(t, x) = \frac{t^2}{x} - t^3 x = c, \quad c \in \mathbb{R}$$

$\frac{1}{\mu} = 0, \quad \frac{x^2}{t} = 0 \Rightarrow x = 0$

$\{x = 0\} \cap G = \phi$ $\lightning$

$0 \cdot dt + \dots d0 = 0$ ✓

кроз $(2,1)$: $t = 2$

$x = 1$

$$\frac{2^2}{1} - 2^3 \cdot 1 = 4 - 8 = -4 = c$$

ПР: $\frac{t^2}{x} - t^3 x = -4.$