2 Хомогена

$$x' = f\left(\frac{x}{t}\right), \quad f \in C(a,b)$$

смена: $x(t) \rightsquigarrow y(t)$

$$\left.y(t) = \frac{x(t)}{t} \Rightarrow x = yt \,/'_t \Rightarrow x' = (yt)' = y'\cdot t + y \cdot \underset{(t)'_t = 1}{1} = y + y't\right\} \rightsquigarrow \text{РП}$$

① а) $x' = e^{\frac{x}{t}} + \frac{x}{t}$

б) $x' = -\frac{x^2+t^2}{2xt}$

в) $x' = \frac{x^2-2xt-t^2}{x^2+2xt-t^2}$

г) $t \sin\frac{x}{t} \cdot x' = x \cdot \sin\frac{x}{t} + t$

г) $/ : t \sin\frac{x}{t}, \quad t \neq 0$

$$x' = \frac{x}{t} + \frac{1}{\sin\frac{x}{t}}, \quad \frac{x}{t} = y$$

$$x = yt \Rightarrow x' = y't + y$$

$$y't + \cancel{y} = \cancel{y} + \frac{1}{\sin y}$$

$$y' \sin y = \frac{1}{t} \;\Big/ \int dt \longrightarrow \int y' \sin y \, dt = \int \sin y \, (\underbrace{y' dt}_{dy = y'dt}) = \int \sin y \, dy$$

$$\int \sin y \, dy = \int \frac{dt}{t}$$

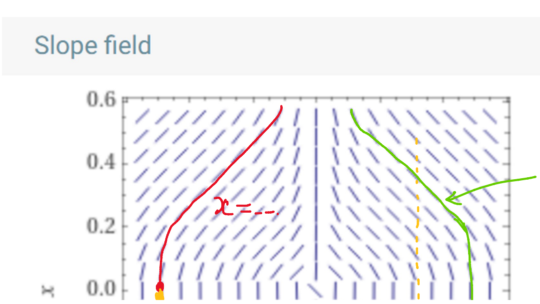
$$-\cos y = \ln|t| + c, \quad c \in \mathbb{R}$$

$$-\cos\frac{x}{t} = \ln|t| + c$$ ← имплицитно задато решење

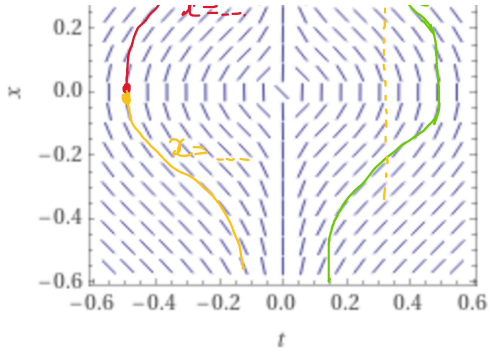
$$\frac{x}{t} = \arccos(-\ln|t| - c)$$

$$x = t \arccos(-\ln|t| - c)$$ ← експлицитно -||- $x(t) = \ldots$

б) $x' = -\frac{x^2+t^2}{2xt}$



← интегрална крива (није ф-ја)



(није ф-ја)

$\boxed{3}$ $x' = f\left(\dfrac{a_1 t + b_1 x + c_1}{a_2 t + b_2 x + c_2}\right)$, $f \in C(a,b)$

$\rightsquigarrow$ РП $\vee$ ХОМ $\vee$ $\boxed{1}$

1° $c_1 = c_2 = 0$: $\quad x' = f\left(\dfrac{a_1 t + b_1 x}{a_2 t + b_2 x}\right) = f\left(\dfrac{a_1 + b_1 \frac{x}{t}}{a_2 + b_2 \frac{x}{t}}\right) = g\left(\dfrac{x}{t}\right) \to$ ХОМ

2° $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \neq 0$

$x(t) \rightsquigarrow v(u)$

$x = v + \alpha \qquad \alpha, \beta \in \mathbb{R}$

$t = u + \beta$

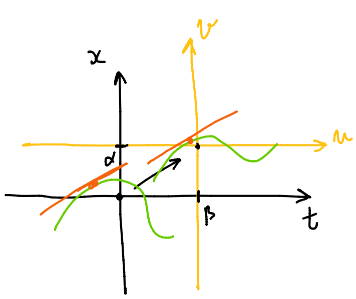
$a_1 t + b_1 x + c_1 = a_1(u+\beta) + b_1(v+\alpha) + c_1 = a_1 u + b_1 v + \boxed{(a_1\beta + b_1\alpha + c_1)} = 0$

$a_2 t + b_2 x + c_2 = \ldots \qquad = a_2 u + b_2 v + \boxed{(a_2\beta + b_2\alpha + c_2)} = 0$

α, β бирамо као реш. сис.:

$$\begin{aligned} a_1\beta + b_1\alpha &= -c_1 \\ a_2\beta + b_2\alpha &= -c_2 \end{aligned} \rightsquigarrow \exists_1\ \alpha, \beta$$

($\det \neq 0$, Крамер)



$\dfrac{dx}{dt} \overset{?}{=} \dfrac{dv}{du}$

$$v' = \frac{dv}{du} = \underbrace{\overset{1}{\frac{dv}{dx}} \cdot \frac{dx}{dt}}_{\frac{dv}{dt}} \cdot \overset{1}{\frac{dt}{du}} = \frac{dx}{dt} = x'$$

$\dfrac{dv}{dx} = \dfrac{d(x-\alpha)}{dx} = 1$

$\dfrac{dt}{du} = \dfrac{d(u+\beta)}{du} = 1$

$$v' = f\left(\frac{a_1 u + b_1 v}{a_2 u + b_2 v}\right) \longrightarrow 1°$$

3° $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$

$$a_1 b_2 = a_2 b_1$$

3.1° ако је неки од a_1, a_2, b_1, b_2 нула

нпр. $a_1 = 0 \Rightarrow a_2 = 0 \lor b_1 = 0$

$a_1 = a_2 = 0$: $\quad x' = f\left(\dfrac{b_1 x + c_1}{b_2 x + c_2}\right) \rightsquigarrow$ РП

$a_1 = b_1 = 0$: $\quad x' = f\left(\dfrac{c_1}{\underbrace{a_2 t + b_2 x + c_2}_{y(t)}}\right) \rightsquigarrow \boxed{1}$

3.2° $\dfrac{b_2}{b_1} = \dfrac{a_2}{a_1} = k$

$$\frac{a_1 t + b_1 x + c_1}{a_2 t + b_2 x + c_2} = \frac{a_1 t + b_1 x + c_1}{k a_1 t + k b_1 x + c_2} = \frac{\boxed{a_1 t + b_1 x} + c_1}{k\underbrace{\boxed{(a_1 t + b_1 x)}}_{y(t)} + c_2} \rightsquigarrow \boxed{1}$$

(2) а) $(x + 2t - 2)x' = x - t - 1$

б) $x' = \dfrac{t + x + 4}{t + x - 6}$

а) $x' = \dfrac{x - t - 1}{x + 2t - 2} \leftarrow 0?$

$x + 2t - 2 = 0?$

$x = -2t + 2$

$0 \cdot (-2) = -3t + 1 \quad \times$

$$\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = 2 - (-1) = 3 \neq 0$$

$x = v + \alpha$

$t = u + \beta$

$$\left.\begin{aligned} \alpha - \beta - 1 &= 0 \\ \alpha + 2\beta - 2 &= 0 \end{aligned}\right\} -$$

$-3\beta + 1 = 0$

$\beta = \frac{1}{3}, \ \alpha = \frac{4}{3}$

$$v' = \frac{dv}{du} = \underbrace{\frac{dv}{dx}}_{=1} \cdot \frac{dx}{dt} \cdot \underbrace{\frac{dt}{du}}_{=1} = x'$$

$$v' = \frac{\left(v + \frac{4}{3}\right) - \left(u + \frac{1}{3}\right) - 1}{\left(v + \frac{4}{3}\right) + 2\left(u + \frac{1}{3}\right) - 2} = \frac{v - u}{v + 2u} = \frac{\frac{v}{u} - 1}{\frac{v}{u} + 2} \quad \text{(хом)}$$

$$\frac{v(u)}{u} = w(u) \Rightarrow v = uw \ /' \Rightarrow v' = w + u \cdot w'$$

$$w + u\cdot w' = \frac{w-1}{w+2}$$

$$u\cdot w' = \frac{w-1}{w+2} - w = \frac{-w^2-w-1}{w+2}$$

$$\frac{w'(w+2)}{w^2+w+1} = -\frac{1}{u} \Big/ \int du$$

0? ⟶ w^2+w+1

⋱

$w^2+w+1=0$? ✗

$w^2+w+1 = \left(w+\frac{1}{2}\right)^2+\frac{3}{4} > 0$

$\Downarrow$

$w(u) \rightsquigarrow v(u) \longrightarrow x(t)$

б) $\begin{vmatrix}1 & 1\\ 1 & 1\end{vmatrix} = 0$

$$x' = \frac{\boxed{x+t}-4}{\boxed{x+t}-6} \rightsquigarrow \boxed{1}$$

$y(t)$

$x+t-6=0$? не мора

4 Линеарна ДЈ 1. реда

$p, q : (a,b) \to \mathbb{R}$ непр.

$$x' + p(t)x = q(t)$$

$\xrightarrow{\text{решавање}}$ $x(t) = e^{-\int p\,dt}\cdot\left(c + \int e^{\int p\,dt}\cdot q\,dt\right),\ c\in\mathbb{R}$

$\int p\,dt = \int_{t_0}^{t} p(u)\,du$

↳ исто на оба места

3 а) $tx' - x = t^3$

б) $x' + x = \frac{1}{1+e^{2t}}$

в) $x' - 2xt = 6te^{t^2}$

г) $tx' + ax + t^n = 0$, $a\in\mathbb{R}, n\in\mathbb{N}$

$q(t)\equiv 0 \to$ хомогена линеарна дј. 1. реда

$q(t)\not\equiv 0 \to$ нехомогена -||-

а) $/ : t$

$$x' \underbrace{-\frac{1}{t}}_{p(t)} x = \underbrace{t^2}_{q(t)}$$

$$\frac{1}{|t|} = \frac{1}{\operatorname{sgn} t\cdot t} = \frac{\operatorname{sgn} t}{t}$$

$$\int p\,dt = \int -\frac{dt}{t} = -\ln|t|$$

$$\int e^{\int p\,dt}\cdot q\,dt = \int e^{-\ln|t|}\cdot t^2\,dt = \int\frac{1}{|t|}\cdot t^2\,dt = \int \underbrace{\operatorname{sgn} t}_{const}\cdot t\,dt = \operatorname{sgn} t\cdot\frac{t^2}{2} = \frac{t|t|}{2}$$

$$x(t) = e^{\ln|t|}\cdot\left(c+\frac{t|t|}{2}\right) = \underbrace{|t|\cdot c}_{c_1 t} + |t|\cdot\frac{t|t|}{2} = \underbrace{c_1 t}_{\text{ОР хомогене}} + \underbrace{\frac{t^3}{2}}_{\text{ПР нехомогене}}$$

$c\in\mathbb{R}$ $\qquad$ $c_1\in\mathbb{R}$

ОР хомогене $g=0$

(4) Наћи реш. дј. $x'\sin t - x\cos t = -\frac{\sin^2 t}{t^2}$ за које важи $\lim\limits_{t\to\infty} x(t)=0$

I начин: по формули

II начин: $(x\sin t)' = x'\sin t + x\cdot\cos t$ $\quad$ −?

$$(fg)' = f'g + fg'$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$/ : \sin^2 t$

$$\frac{x'\sin t - x\cos t}{\sin^2 t} = -\frac{1}{t^2}$$

$$\left(\frac{x}{\sin t}\right)' = -\frac{1}{t^2} \quad \Big/ \int dt$$

$$\frac{x}{\sin t} = \frac{1}{t} + c$$

$$x = \frac{\sin t}{t} + c\sin t, \quad c\in\mathbb{R}$$

$$\lim_{t\to\infty}\left(\frac{\sin t}{t} + c\sin t\right) = 0$$

није Кошијев услов $x(t_0)=x_0$

$$\left|\frac{\sin t}{t}\right| \le \frac{|\sin t|}{t} \le \frac{1}{t}, \quad t\to\infty$$

$$\frac{\sin t}{t} \xrightarrow[t\to\infty]{} 0$$

$$\Downarrow$$

$$\lim_{t\to\infty} c\cdot\sin t = 0 \Rightarrow c = 0$$

ПР: $x(t) = \frac{\sin t}{t}$

(5) Бернулијева ДЈ

$$x' + p(t)x = g(t)\cdot x^{\alpha}, \quad \alpha\in\mathbb{R} \qquad p,g : (a,b)\to\mathbb{R} \text{ непр.}$$

$\alpha\in\{0,1\} \to$ линеарна

$\alpha\notin\{0,1\}$ надаље

смена: $y(t) = x(t)^{1-\alpha}$

$$y' = (x^{1-\alpha})' = (1-\alpha)\cdot x^{-\alpha}\cdot x' \Rightarrow x'x^{-\alpha} = \frac{y'}{1-\alpha}$$

$\cdot x^{-\alpha}$

$$\underbrace{x'x^{-\alpha}}_{y'} + p(t)\cdot\underbrace{x^{1-\alpha}}_{y} = g(t) \quad /\cdot(1-\alpha)$$

$$\underbrace{x'x^{-\alpha}}_{\frac{y'}{1-\alpha}} + p(t)\cdot \underbrace{x^{1-\alpha}}_{y} = q(t) \quad /\cdot(1-\alpha)$$

$$y' + (1-\alpha)p(t)y = (1-\alpha)q(t) \quad \to \text{лин.}$$

(5) а) $x' = \frac{x}{t} - x^2$

б) $x' + \frac{x}{t} = x^2 \frac{\ln t}{t}$

в) $tx' - 2t\sqrt{x} = 4x$

в) $/:t$

$$x' - 2\sqrt{x} = 4\frac{x}{t}$$

$$x' - \frac{4}{t}\cdot x = 2\sqrt{x} \qquad \sqrt{x} = x^{\alpha} \Rightarrow \alpha = \frac{1}{2}$$

$$y = x^{1-\frac{1}{2}} = x^{\frac{1}{2}}$$

$$x^{-\frac{1}{2}} \quad y' = \frac{1}{2}x^{-\frac{1}{2}}x' \Rightarrow x^{-\frac{1}{2}}x' = 2y'$$

$$\underbrace{x'x^{-\frac{1}{2}}}_{2y'} - \frac{4}{t}\cdot \underbrace{x^{\frac{1}{2}}}_{y} = 2 \quad /:2$$

$$y' - \frac{2}{t}y = 1 \to \text{лин.}$$

$$p(t) = -\frac{2}{t} \qquad \int p\,dt = \int -\frac{2dt}{t} = -2\ln|t|$$

$$q(t) = 1 \qquad \int e^{\int p\,dt}\cdot q\,dt = \int |t|^{-2}\cdot 1\,dt = \int \frac{dt}{t^2} = -\frac{1}{t}$$

$$y(t) = t^2\left(c - \frac{1}{t}\right) = ct^2 - t \quad , c\in\mathbb{R}$$

$$y = x^{1/2} \Rightarrow x = y^2 = (ct^2 - t)^2$$

а) $x' = \frac{x}{t} - x^2$

$$x' - \frac{x}{t} = -x^2$$

$$x^2 = x^{\alpha} \Rightarrow \alpha = 2$$

$$y = x^{1-\alpha} = x^{1-2} = x^{-1} = \frac{1}{x} \ldots$$