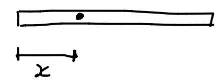
Парцијалне ДЈ 1. реда

ОДЈ $\quad x(t) \to$ еволуција у времену, $\quad F = \ddot{x} \quad$ — II Њ. закон

ПДЈ $\quad u(x_1,\dots,x_n) = ? \to$ еволуцију у простору и времену, нпр. $u_t = u_{xx}$

јна провођења топлоте

$u(x,t)$ — интензитет топлоте



t - време

нотација: $u_t = u'_t = \frac{\partial u}{\partial t}$

$$u_{xx} = u''_{xx} = \frac{\partial^2 u}{\partial x^2}$$

$$v_{xy} = v''_{xy} = \frac{\partial^2 v}{\partial x \partial y}$$

$u_t = u_{xx}$ → 2. реда

1. реда

1\. реда $\rightsquigarrow F\left(x_1,\dots,x_n,u,\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n}\right) = 0$

→ квазилинеарна (КЛ): $\displaystyle\sum_{j=0}^{n} a_j(x_1,\dots,x_n,u)\frac{\partial u}{\partial x_j} = c(x_1,\dots,x_n,u)$

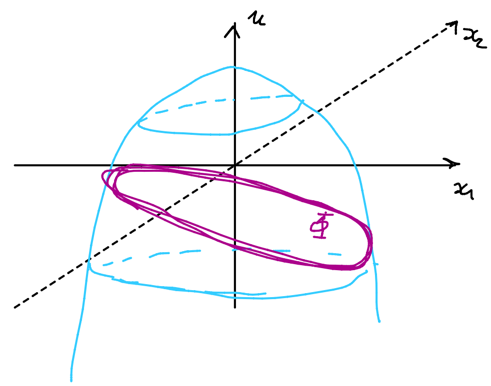
$\left[\text{линеарна: } \displaystyle\sum_{j=0}^{n} a_j(x_1,\dots,x_n)\frac{\partial u}{\partial x_j} = c(x_1,\dots,x_n)\right]$

→ хомогена линеарна (ХЛ): $\displaystyle\sum_{j=0}^{n} a_j(x_1,\dots,x_n)\frac{\partial u}{\partial x_j} = 0 \qquad (c=0)$

Кошијев проблем: наћи решење које садржи задату фигуру Φ

$\Phi \subseteq \Gamma(u)$

↳ график

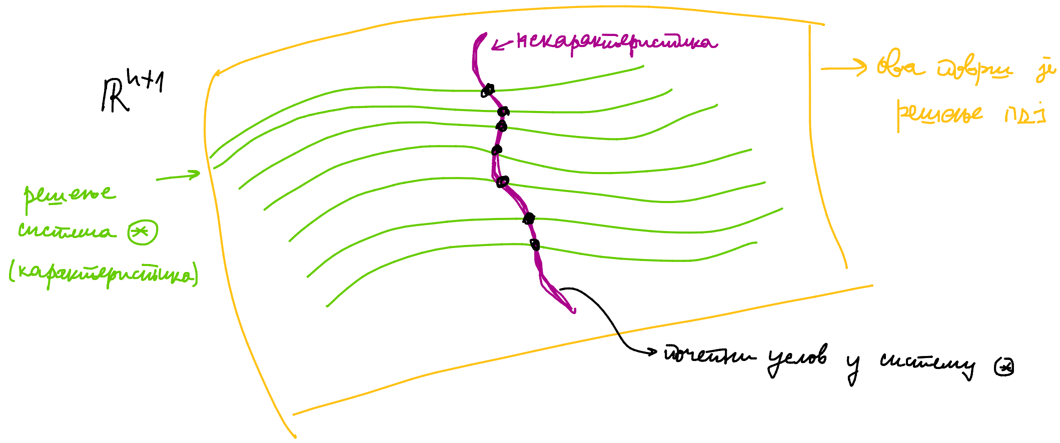


Метода карактеристика (за Кошијев проблем)

(КЛ) ⇒ систем карактеристика (*)

$\displaystyle\sum_{j=0}^{n} a_j(x_1,\dots,x_n,u)\frac{\partial u}{\partial x_j} = c(x_1,\dots,x_n,u) \Rightarrow \qquad x_j'(t) = a_j(x_1,\dots,x_n,u), \ \forall j$

$$\sum_{j=0}^{n} a_j(x_1,\dots,x_n,u)\frac{\partial u}{\partial x_j} = c(x_1,\dots,x_n,u) \Rightarrow \left.\begin{aligned} &x_j'(t) = a_j(x_1,\dots,x_n,u), \ \forall j \\ &u'(t) = c(x_1,\dots,x_n,u) \end{aligned}\right\} (*)$$



① $u_x' + u_y' + 2u = 1+u^2$, $u(\sin x, x+x^2) = x$ ↳ Кошијев услов

$u(x,y)$

$$\underbrace{1}_{a_1}\cdot u_x' + \underbrace{1}_{a_2} u_y' = \underbrace{1+u^2-2u}_{c} \Rightarrow \left.\begin{aligned} &x'(t) = 1 \\ &y'(t) = 1 \\ &u'(t) = 1+u^2-2u \end{aligned}\right\}$$

$x' = 1 \Rightarrow x(t) = t + c_1$

$y' = 1 \Rightarrow y(t) = t + c_2$

$u' = (u-1)^2 \Rightarrow \frac{u'}{(u-1)^2} = 1$ (РП) $/\int$

$$\int \frac{du}{(u-1)^2} = t + c_3$$

$$-\frac{1}{u-1} = t + c_3 \Rightarrow 1-u = \frac{1}{t+c_3} \Rightarrow u = 1 - \frac{1}{t+c_3} = \frac{t+c_3-1}{t+c_3}$$

$u(\underbrace{\sin x}_{x}, \underbrace{x+x^2}_{y}) = \underbrace{x}_{u}$ → крива у $\mathbb{R}^3$ (xyu)

$\gamma(s) = (\sin s, s+s^2, s)$ → некарактеристика $\in \Gamma(u) \subseteq \mathbb{R}^3$

$c_1, c_2, c_3 \rightsquigarrow c_1(s), c_2(s), c_3(s)$

$$x(t,s) = t + c_1(s)$$
$$y(t,s) = t + c_2(s)$$
$$u(t,s) = \frac{t + c_3(s) - 1}{t + c_3(s)}$$

почетни услов: $x(0,s) = x_0(s) = \sin s$

$y(0,s) = y_0(s) = s + s^2$

$u(0,s) = u_0(s) = s$

$$c_1(s) = \sin s$$
$$c_2(s) = s + s^2$$
$$\frac{c_3(s) - 1}{c_3(s)} = s \Rightarrow c_3(s) = \frac{1}{1-s}$$

површ (параметарски задата — s и t):

$$(x(t,s), y(t,s), u(t,s)) = \left(t + \sin s,\ t + s + s^2,\ \frac{t + \frac{1}{1-s} - 1}{t + \frac{1}{1-s}}\right) \checkmark \subseteq \mathbb{R}^3$$

(2) $(y+u)u'_x + y u'_y = x - y$, $u|_{y=1} = x+1$

$u(x,y)$

$u(x,1) = x+1$, $\gamma(s) = (s, 1, s+1) \subseteq \Gamma(u) \subseteq \mathbb{R}^3$

$$x' = y + u$$
$$y' = y$$
$$u' = x - y$$

$x_0(s) = s$

$y_0(s) = 1$

$u_0(s) = s+1$

$$X = \begin{bmatrix} x \\ y \\ u \end{bmatrix}, \quad X' = AX$$
$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

```
>> A = [0 1 1; 0 1 0; 1 -1 0]

A =

     0     1     1
     0     1     0
     1    -1     0

>> [P D] = jordan(A)

P =

    -1     1     1
     0     1     0
     1     0     1


D =

    -1     0     0
     0     1     0
     0     0     1
```

$$A = PDP^{-1}$$
$$e^{tA} = Pe^{tD}P^{-1}$$

$$\text{OP}: X(t) = e^{tA} \cdot c = P \cdot e^{tD} \cdot c_1 = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{-t} & & \\ & e^{t} & \\ & & e^{t} \end{bmatrix} \cdot c_1 = \begin{bmatrix} -e^{-t} & e^{t} & e^{t} \\ 0 & e^{t} & 0 \\ e^{-t} & 0 & e^{t} \end{bmatrix} \cdot c_1$$

$$X(t,s) = [\ldots] \cdot c_1(s)$$

$$X(0,s) = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} c(s) \qquad X(0,s) = \begin{bmatrix} x_0(s) \\ y_0(s) \end{bmatrix} = \begin{bmatrix} s \\ 1 \end{bmatrix}$$

0	1	0
0	0	1

$$X(0,s)=\begin{bmatrix}-1&1&1\\0&1&0\\1&0&1\end{bmatrix}\cdot G(s) \qquad X(0,s)=\begin{bmatrix}x_0(s)\\y_0(s)\\u_0(s)\end{bmatrix}=\begin{bmatrix}s\\1\\s+1\end{bmatrix}$$

$$\Rightarrow G(s)=\begin{bmatrix}-1&1&1\\0&1&0\\1&0&1\end{bmatrix}^{-1}\cdot\begin{bmatrix}s\\1\\s+1\end{bmatrix}$$

```
>> inv([-1 1 1; 0 1 0; 1 0 1])

ans =

   -0.5000    0.5000    0.5000
         0    1.0000         0
    0.5000   -0.5000    0.5000
```

$$=\frac{1}{2}\cdot\begin{bmatrix}-1&1&1\\0&2&0\\1&-1&1\end{bmatrix}\cdot\begin{bmatrix}s\\1\\s+1\end{bmatrix}=\begin{bmatrix}1\\1\\s\end{bmatrix}$$

$$X(t,s)=\begin{bmatrix}-e^{-t}&e^{t}&e^{t}\\0&e^{t}&0\\e^{-t}&0&e^{t}\end{bmatrix}\cdot\begin{bmatrix}1\\1\\s\end{bmatrix}=\begin{bmatrix}\overset{=x}{e^{t}(1+s)-e^{-t}}\\e^{t}=y\\\underset{=u}{e^{-t}+se^{t}}\end{bmatrix}$$

решење: $(x,y,u)=\left(e^{t}(1+s)-e^{-t},\ e^{t},\ e^{-t}+se^{t}\right)$ - површ у $\mathbb{R}^3$

смене: $e^{t}=y \Rightarrow t=\ln y$

$$e^{-t}=e^{-\ln y}=\frac{1}{y}$$

$$x=e^{t}(1+s)-e^{-t}=y(1+s)-\frac{1}{y} \Rightarrow s=\frac{x}{y}+\frac{1}{y^2}-1$$

$$u(x,y)=e^{-t}+se^{t}=\frac{1}{y}+\left(\frac{x}{y}+\frac{1}{y^2}-1\right)\cdot y=\underline{\underline{x-y+\frac{2}{y}}}.$$

провера: $(y+u)u'_x+yu'_y=x-y$, $u|_{y=1}=x+1$

$$\frac{\partial u}{\partial x}=1$$

$$\frac{\partial u}{\partial y}=-1-\frac{2}{y^2}$$

$$u(x,1)=x-1+\frac{2}{1}=x+1 \ \checkmark$$

$$\left(\cancel{y}+x-\cancel{y}+\frac{2}{y}\right)\cdot 1+y\cdot\left(-1-\frac{2}{y^2}\right)=x-y$$

$$x+\frac{2}{y}-y-\frac{2}{y}=x-y \ \checkmark$$

3 $\quad \underline{y}\frac{\partial u}{\partial x}-x\frac{\partial u}{\partial y}=0 \qquad (\underline{s},\underline{s},\underline{s}^4)\subseteq\Gamma(u)$

$(x,1)$, али посматрано као $(s,1)$

(КП), али посматрано као (КП)

$$\left.\begin{aligned} a_1(x,y,u) &= y \\ a_2(x,y,u) &= -x \\ c(x,y,u) &= 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} x' &= y \\ y' &= -x \\ u' &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x &= c_1 \cos t + c_2 \sin t \\ y &= c_2 \cos t - c_1 \sin t \\ u &= c_3 \end{aligned} \qquad \begin{aligned} x_0(s) &= s \\ y_0(s) &= s \\ u_0(s) &= s^4 \end{aligned}$$

$$\Rightarrow \begin{aligned} c_1 &= s \\ c_2 &= s \\ c_3 &= s^4 \end{aligned}$$

површ: $(s(\cos t+\sin t),\ s(\cos t-\sin t),\ s^4)$

Тсп.: $x^2+y^2 = s^2(\underbrace{\cos^2 t+\sin^2 t}_{1} + 2\cos t\sin t + \underbrace{\cos^2 t+\sin^2 t}_{1} - 2\cos t \sin t) = 2s^2$

$$\frac{(x^2+y^2)^2}{4} = s^4 = u$$

приметимо: $x = c_1\cos t + c_2 \sin t$

$y = c_2 \cos t - c_1 \sin t$

$x^2+y^2 = c_1^2+c_2^2 = const$ $\Rightarrow$ x^2+y^2 је први интеграл система (*)

тврдимо да је $u=\varphi(x^2+y^2)$, $\varphi\in C^1(\mathbb{R})$ решење ПДЈ:

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0, \qquad \frac{\partial u}{\partial x} = \varphi'(x^2+y^2)\cdot 2x$$

$$\frac{\partial u}{\partial y} = \varphi'(x^2+y^2)\cdot 2y$$

$$y\cdot\varphi'(x^2+y^2)\cdot 2x - x\cdot \varphi'(x^2+y^2)\cdot 2y = 0 \ \checkmark$$

Метода првих интеграла

идеја: у систему (*) наћи $\boxed{\text{скупове}}$ где је решење ПДЈ константно

↳ први интеграли система (*) $\psi_1, \dots, \psi_{n-1}$

(КП) $\Rightarrow$ (*), $u(x_1,\dots,x_n)$ се тражи ⇝ сис. кар. је реда $n+1$ ⇝ потребно је n првих интеграла $\psi_1,\dots,\psi_n$. ОР: $\boxed{\varphi(\psi_1,\dots,\psi_n)=0}$, $\varphi\in C^1(\mathbb{R}^n)$

интеграла $\psi_1,\dots,\psi_n$. ОР: $\boxed{\Phi(\psi_1,\dots,\psi_n)=0}$, $\Phi\in C^1(\mathbb{R}^n)$

→ имплицитно задато

(КЛ) $\Rightarrow$ (*), $u(x_1,\dots,x_n)$ се тражи ⇝ сис. кар. је реда n ⇝ потребно је $n-1$ првих интеграла $\psi_1,\dots,\psi_{n-1}$. ОР: $\boxed{u=\Phi(\psi_1,\dots,\psi_{n-1})}$, $\Phi\in C^1(\mathbb{R}^{n-1})$.

→ експл. задато

(*) сис. кар. за (КЛ): $\sum_{j=0}^{n} a_j(x_1,\dots,x_n)\frac{\partial u}{\partial x_j}=0 \quad\Rightarrow\quad x_j'(t)=a_j(x_1,\dots,x_n),\ \forall j$

4) $(x^2-y^2-z^2)\frac{\partial z}{\partial x}+2xy\frac{\partial z}{\partial y}=2xz$, наћи ОР

(КЛ)

$z(x,y)$ → 2 прва интеграла

$$x'=x^2-y^2-z^2$$
$$y'=2xy$$
$$z'=2xz$$

$$\int\frac{y'}{y}dt=\int\frac{dy}{y}=\ln|y|+c$$
$$(\ln|y|)'=\frac{1}{|y|}\cdot\operatorname{sgn}(y)\cdot y'=\frac{y'}{y}$$

$$\frac{y'}{z'}=\frac{y}{z}\Rightarrow\frac{y'}{y}=\frac{z'}{z}\ \Big/\int dt$$

$$\Rightarrow \ln|y|=\ln|z|+\tilde{C}_1$$

$$\ln|y|-\ln|z|=\tilde{C}_1$$

то је 1. интеграл!

$$\ln\left|\frac{y}{z}\right|=\tilde{C}_1 \qquad (C_1=\pm e^{\tilde{C}_1})$$

$$\frac{y}{z}=C_1\Rightarrow\psi_1(x,y,z)=\frac{y}{z}.$$

$y=C_1z \to x'=x^2-y^2-z^2$

$$\left.\begin{aligned}x'&=x^2-(C_1^2+1)z^2\\ z'&=2xz\end{aligned}\right\}$$

$$\frac{x'}{z'}=\frac{x^2-(C_1^2+1)z^2}{2xz}=\frac{x}{2z}-\frac{C_1^2+1}{2}\cdot\frac{z}{x}$$

$$\frac{dx}{dz}=\frac{x}{2z}-\frac{C_1^2+1}{2}\cdot\frac{z}{x}\ \to x^{-1}$$

$$\frac{x'}{z'}=\frac{\frac{dx}{\cancel{dt}}}{\frac{dz}{\cancel{dt}}}\ \text{"="}\ \frac{dx}{dz}$$

→ смена $x(t),z(t)$ ⇝ $x(z)$

$$\frac{dx}{dz}=\frac{dx}{dt}\cdot\frac{dt}{dz},\qquad \frac{dt}{dz}=\left(\frac{dz}{dt}\right)^{-1}$$

$\frac{dx}{dz} = \frac{\cdot}{2z} - \frac{\cdot}{2}\cdot x \to x^{-1}$

$\frac{dx}{dz} = \frac{dx}{dt}\cdot\left(\frac{dt}{dz}\right)$, $\left(\frac{dz}{dt}\right)^{-1}$

$\rightsquigarrow$ Бернулијева за $\alpha = -1$

$$u(z) = x(z)^{1-\alpha} = x(z)^2$$

$$x(z)^2 = u(z) = cz - z^2(1+c_1^2), \quad c_1 = \frac{y}{z}$$

$$\Rightarrow x^2 = cz - z^2\left(1+\frac{y^2}{z^2}\right) = cz - y^2 - z^2$$

$$c = \frac{x^2+y^2+z^2}{z} = \psi_2(x,y,z)$$

ОР: $\varphi(\psi_1, \psi_2) = 0$, $\varphi \in C^1(\mathbb{R}^2)$

$$\varphi\left(\frac{y}{z}, \frac{x^2+y^2+z^2}{z}\right) = 0$$

Кошијев проблем $\rightsquigarrow$ налазимо φ!

5) $(4y-3z)\frac{\partial u}{\partial x} + (4x-2z)\frac{\partial u}{\partial y} + (2y-3x)\frac{\partial u}{\partial z} = 0$ са условом $u(x,y,1) = (x+2)^2-(y+3)^2$

$u(x,y,z)$, $(X\cap)$

2 прва инт.

$$x' = 4y-3z$$
$$y' = 4x-2z$$
$$z' = 2y-3x$$

домаћи: методом карактеристика $X' = AX$, e^{tA} ...

идеја: линеарне комбинације

$\alpha, \beta \in \mathbb{R}$

$$\alpha\cdot(4y-3z) + \beta\cdot(4x-2z) = 2y-3x$$

$$x\cdot(4\beta) + y\cdot(4\alpha) + z\cdot\underbrace{(-3\alpha-2\beta)}_{=0} = -3x+2y$$

$$\beta = -\frac{3}{4}$$
$$\alpha = \frac{1}{2}$$

$\int x'\,dt = x + c$

$$\frac{1}{2}\cdot x' - \frac{3}{4}\cdot y' = z' \quad /\int$$

$$\frac{1}{2}\cdot x' - \frac{3}{4}\cdot y' = z' \;/\int dt$$

$$\frac{x}{2} - \frac{3}{4}y = z + c \;/\cdot 4$$

$$2x - 3y - 4z = c_1 = \psi_1(x,y,z)$$

другачију лин. комб: $\alpha\cdot x\cdot(4y-3z) + \beta\cdot y\cdot(4x-2z) = z\cdot(2y-3x)$

$$xy\cdot(\underbrace{4\alpha+4\beta}_{=0}) + yz(-2\beta) + zx(-3\alpha) = 2yz - 3zx$$

$$\alpha = 1$$
$$\beta = -1$$

$$x\cdot x' - y\cdot y' = z\cdot z' \;/\int dt$$

$$\frac{x^2}{2} - \frac{y^2}{2} = \frac{z^2}{2} + c \;/\cdot 2$$

$$\psi_2(x,y,z) = x^2 - y^2 - z^2$$

$$\int x\cdot x'\,dt = \frac{x^2}{2} + c$$

$$\left(\frac{x^2}{2}\right)' = \frac{1}{2}\cdot 2\cdot x\cdot x' = xx'$$

$$\boxed{\cancel{\int x\cdot y'\,dt = x\cdot y + c}}$$

$$(xy)' = x'y + y'x$$

$$\int (x'y + xy')\,dt = xy + c$$

OP: $u = \varphi(\psi_1, \psi_2)$, $\varphi \in C^1(\mathbb{R}^2)$

$$u(x,y,z) = \varphi(2x - 3y - 4z,\; x^2 - y^2 - z^2)$$

усл. Кошијев: $u(x,y,1) = (x+2)^2 - (y+3)^2$

$z=1$ $\quad u(x,y,1) = \varphi(2x-3y-4,\; x^2-y^2-1)$

$$\| $$

$$(x+2)^2 - (y+3)^2$$

$$\|$$

$$x^2 - y^2 + 4x - 6y - 5$$

$$\rightarrow \varphi(p_1, p_2) = 2p_1 + p_2 + 4$$

Кошијево реш: $u(x,y,z) = 2\psi_1 + \psi_2 + 4 = 4x - 6y - 8z + x^2 - y^2 - z^2 + 4 =$

$$= (x+2)^2 - (y+3)^2 - (z+4)^2 + 25$$

⑥ $x(y^2 - z^2)\dfrac{\partial u}{\partial x} - y(x^2+z^2)\dfrac{\partial u}{\partial y} + z(x^2+y^2)\dfrac{\partial u}{\partial z} = 0$, $\quad u|_{x=1} = (y+z)^2$

$(x\wedge)$

$u(x,y,z)$ ~> 2 прва инт.

$$u(1,y,z) = (y+z)^2$$

$$(1, y, z, (y+z)^2) \subseteq \Gamma(u) \subseteq \mathbb{R}^4$$

$$x' = x(y^2 - z^2)$$

$$y' = -y(x^2+z^2)$$

$$x' = x(y^2 - z^2)$$
$$y' = -y(x^2 + z^2)$$
$$z' = z(x^2 + y^2)$$

- $\alpha x(y^2 - z^2) - \beta y(x^2 + z^2) = z(x^2 + y^2)$

- $\alpha x \cdot x(y^2 - z^2) - \beta y \cdot y(x^2 + z^2) = z \cdot z(x^2 + y^2)$

$$\alpha = \beta = -1$$
$$-xx' - yy' = zz' \quad / \int dt$$
$$-\frac{x^2}{2} - \frac{y^2}{2} = \frac{z^2}{2} + c$$
$$\psi_1(x, y, z) = x^2 + y^2 + z^2$$

- $\frac{\alpha}{x} \cdot x(y^2 - z^2) - \frac{\beta}{y} \cdot y(x^2 + z^2) = \frac{1}{z} \cdot z(x^2 + y^2)$

$$\alpha(y^2 - z^2) - \beta(x^2 + z^2) = x^2 + y^2$$

$$\int \frac{x'}{x} dt = \ln|x| + c$$

$$\alpha = 1, \ \beta = -1$$
$$\frac{x'}{x} - \frac{y'}{y} = \frac{z'}{z} \quad / \int dt$$
$$\ln|x| - \ln|y| = \ln|z| + c$$
$$\hookrightarrow \ln\left|\frac{x}{yz}\right| = c \rightsquigarrow \psi_2(x, y, z) = \frac{x}{yz}$$

ОР: $u = \varphi(\psi_1, \psi_2), \ \varphi \in C^1(\mathbb{R}^2)$

$u|_{x=1} = (y+z)^2$

$$\underset{\substack{\| \\ (y+z)^2}}{u(1, y, z)} = \varphi\left(\psi_1|_{x=1}, \ \psi_2|_{x=1}\right) = \varphi\left(1 + y^2 + z^2, \ \frac{1}{yz}\right)$$

$$(y+z)^2 = \underbrace{(1 + y^2 + z^2)} + 2\frac{1}{\frac{1}{yz}} - 1$$

$$\varphi(p_1, p_2) = p_1 + \frac{2}{p_2} - 1$$

Кошијево реш.: $u = \psi_1 + \frac{2}{\psi_2} - 1 = x^2 + y^2 + z^2 + \frac{2yz}{x} - 1.$