ЛЈВРКК (лин. јне вишег реда са КК)

$$x^{(n)} + a_{n-1}(t)x^{(n-1)} + a_{n-2}(t)x^{(n-2)} + \dots + a_1(t)x' + a_0(t)x = f(t)$$ → лин. јна ВР

$a_{n-1}(t), \dots, a_0(t)$ – константе: ЛЈВРКК

$f \neq 0$: нехомогена

$f = 0$: хомогена

$$x^{(n)} + a_{n-1}x^{(n-1)} + \dots + a_1 x' + a_0 x = f(t)$$

ОР: $x(t) = \underbrace{x_H(t)}_{\text{ОР хомогене}} + \underbrace{x_p(t)}_{\text{ПР нехомогене}}$, ($f=0$, $x_p=0$)

за хомогену

$$\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0 \quad (*)$$ (карактеристична јна)

⇒ n нула (над $\mathbb{C}$)

→ како изгледа база ВП решења x_H? (ВП дим $= n$)

1) $\mu \in \mathbb{R}$ нула $(*)$ реда k:

базни ел. су $e^{\mu t}, te^{\mu t}, \dots, t^{k-1}e^{\mu t}$ (има их k)

2) $\alpha \pm i\beta \in \mathbb{C} \setminus \mathbb{R}$ нула $(*)$ реда k:

базни ел. су $e^{\alpha t}\cos\beta t, e^{\alpha t}\sin\beta t, te^{\alpha t}\cos\beta t, te^{\alpha t}\sin\beta t, \dots, t^{k-1}e^{\alpha t}\cos\beta t, t^{k-1}e^{\alpha t}\sin\beta t$

(2k их је)

1 (хомогене)

а) $x''' - 13x' - 12x = 0$

кар. јна.

$$\lambda^3 - 13\lambda - 12 = 0$$

$\lambda = 1$ ✗

$\lambda = -1$ ✓

$$\lambda^3 - 13\lambda - 12 = (\lambda+1)(\lambda^2 - \lambda - 12)$$

$\qquad = (\lambda-4)(\lambda+3)$

$\lambda_1 = -1 \rightsquigarrow e^{-t}$

$\lambda_2 = -3 \rightsquigarrow e^{-3t}$

$\lambda_3 = 4 \rightsquigarrow e^{4t}$

$\lambda = -1$ ✓ $\qquad (\lambda-4)(\lambda+3) \qquad \lambda_2 = -3 \rightsquigarrow e^{-3t}$

$\lambda_3 = 4 \rightsquigarrow e^{4t}$

$$\text{OP}: \; x(t) = c_1 e^{-t} + c_2 e^{-3t} + c_3 e^{4t}, \quad c_1, c_2, c_3 \in \mathbb{R}$$

б) $x''' - 3x'' + 9x' + 13x = 0$

$\lambda^3 - 3\lambda^2 + 9\lambda + 13 = 0$

$\lambda_1 = -1 \rightsquigarrow e^{-t}$

$$\lambda^3 - 3\lambda^2 + 9\lambda + 13 = (\lambda+1)\underbrace{(\lambda^2 - 4\lambda + 13)}$$

$\lambda^2 - 4\lambda + 13 = 0$

$D = (-4)^2 - 4 \cdot 13 = 16 - 52 = -36$

$\lambda_{2|3} = \dfrac{4 \pm i \cdot 6}{2} = 2 \pm 3i$

$\rightsquigarrow e^{2t}\cos 3t, \; e^{2t}\sin 3t$

$$\text{OP}: \; x(t) = c_1 e^{-t} + c_2 e^{2t}\cos 3t + c_3 e^{2t}\sin 3t, \quad c_i \in \mathbb{R}$$

в) $x''' - 7x'' + 16x' - 12x = 0$

$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$

$\lambda = 1$ ✗

$\lambda = -1$ ✗

$\underline{\underline{\lambda = 2}}$

$(\lambda - 2)(\lambda^2 - 5\lambda + 6)$
$\qquad \parallel$
$(\lambda-2)(\lambda-3)$

$\lambda_1 = \lambda_2 = 2 \rightsquigarrow e^{2t}, te^{2t}$

$\lambda_3 = 3 \rightsquigarrow e^{3t}$

$$\text{OP}: \; x(t) = c_1 e^{2t} + c_2 te^{2t} + c_3 e^{3t}, \quad c_i \in \mathbb{R}$$

г) $x^{(6)} - 4x^{(5)} + 8x^{(4)} - 8x''' + 4x'' = 0$

$\lambda^6 - 4\lambda^5 + 8\lambda^4 - 8\lambda^3 + 4\lambda^2 = 0$

$\lambda^2(\lambda^4 - 4\lambda^3 + 8\lambda^2 - 8\lambda + 4) = 0$
$\qquad \parallel$
$(\lambda^2 + a\lambda + b)(\lambda^2 + c\lambda + d)$

± 1
± 2 ✗
± 4

↳ делује да нема $\mathbb{R}$ нула

$\lambda^3: \quad a + c = -4$

$\lambda^2: \quad b + d + ac = 8$

$\lambda: \quad ad + bc = -8$

1: $bd = 4 \rightsquigarrow (1,4), (2,2), (-2,-2), (-1,-4) \ldots$

$$b = d = 2, \quad a = c = -2 \quad \Rightarrow \quad (\lambda^2 - 2\lambda + 2)^2$$

$$\lambda_1 = \lambda_2 = 0 \rightsquigarrow \underset{=1}{e^{0t}}, \underset{=t}{te^{0t}}$$

$$\lambda_{3/4} = \lambda_{5/6} = 1 \pm i \rightsquigarrow e^t \cos t, e^t \sin t, te^t \cos t, te^t \sin t$$

$$\text{OP}: x(t) = c_1 + c_2 t + c_3 e^t \cos t + c_4 e^t \sin t + c_5 t e^t \cos t + c_6 t e^t \sin t, \quad c_i \in \mathbb{R}$$

Кошијев проблем: $x^{(n)} + a_{n-1}(t) x^{(n-1)} + \ldots + a_0(t) x = f(t)$

услови: $\left.\begin{aligned} x(t_0) &= x_0 \\ x'(t_0) &= x_1 \\ &\vdots \\ x^{(n-1)}(t_0) &= x_{n-1} \end{aligned}\right\}$ n јна

↳ у истој тачки

није КП: $x(0) = x(1) = 0$ ✗

$x(0) = 0, x'(1) = 2$ ✗

② Решити Кошијев проблем: $\overset{3.\text{ реда}}{x'''} + x'' = 0$

$x(0) = 1$
$x'(0) = 0$
$x''(0) = 1$

$$\lambda^3 + \lambda^2 = 0$$
$$\lambda^2(\lambda + 1) = 0$$
$$\lambda_1 = \lambda_2 = 0 \rightsquigarrow 1, t$$
$$\lambda_3 = -1 \rightsquigarrow e^{-t}$$

$$\text{OP}: x(t) = c_1 + c_2 t + c_3 e^{-t}, \quad c_i \in \mathbb{R}$$

$x'(t) = c_2 - c_3 e^{-t}$

$x''(t) = c_3 e^{-t}$

$$\left.\begin{aligned} x(0) = 1 &\Rightarrow c_1 + c_3 = 1 \\ x'(0) = 0 &\Rightarrow c_2 - c_3 = 0 \\ x''(0) = 1 &\Rightarrow c_3 = 1 \end{aligned}\right\}$$

$$c_1 = 0$$
$$c_2 = c_3 = 1$$

$$x_K(t) = t + e^{-t}$$

нехомогени случај: $f \neq 0$, $\quad x^{(n)} + a_{n-1} x^{(n-1)} + \ldots + a_1 x' + a_0 x = f(t)$

P_n – пол. ст. n
Q_m – пол. ст. m

у посебним случајевима: $f(t) = e^{\alpha t} \cdot (P_n(t) \cdot \cos \beta t + Q_m(t) \sin \beta t)$

s – вишеструкост броја $\alpha \pm i\beta$ као решења карақт. јне

$x_p(t) = t^s \cdot e^{\alpha t} (R_k(t) \cos \beta t + T_k(t) \sin \beta t)$

R_k, T_k – пол.

$$x_p(t) = t^{\lambda} \cdot e^{\alpha t} \cdot \left(R_k(t)\cos\beta t + T_k(t)\cdot\sin\beta t\right)$$

R_k, T_k - пол. ст. k

$k = \max\{m, n\}$

(3) a) $x''' - x'' + x' - x = t^2 + t$

$\lambda^3 - \lambda^2 + \lambda - 1 = 0$

$(\lambda - 1)(\lambda^2 + 1) = 0$

$1, \pm i$

ОР: $x(t) = x_H(t) + x_p(t)$

$x_H(t) = c_1 e^t + c_2 \cos t + c_3 \sin t$

$f(t) = t^2 + t$

$\alpha = 0 \quad (e^{\alpha t} = 1)$

$\beta = 0 \quad (\cos\beta t = 1, \sin\beta t = 0)$

$\alpha \pm i\beta = 0 \pm i0 = \underline{\underline{0}}$

$P_n(t) = t^2 + t \quad , n = 2$

$m = 0$ (недостаје) $\quad k = \max\{0, 2\} = 2$

$\lambda = ?$ 0 није у скупу $\{1, \pm i\} \Rightarrow \lambda = 0$

$x_p(t) = R_2(t) = at^2 + bt + c \quad \rightsquigarrow \quad x_p''' - x_p'' + x_p' - x_p = t^2 + t$

$x_p' = 2at + b$

$x_p'' = 2a$

$x_p''' = 0$

$\longrightarrow \quad 0 - 2a + 2at + b - at^2 - bt - c = t^2 + t$

$-2a + b - c = 0$

$2a - b = 1$

$-a = 1$

$a = -1, \ b = -3, \ c = -1$

$x_p(t) = -t^2 - 3t - 1$

б) $x''' - x'' + x' - x = \cos t + 2e^t \longrightarrow$ није у одг. облику

$1, \pm i$

$x(t) = c_1 + c_2\cos t + c_3 \sin t + x_{p_1}(t) + x_{p_2}(t)$

$f_1(t) = \cos t$

$f_2(t) = 2e^t$

лин.

$L(x) = f_1(t) + f_2(t)$

$L(x_{p_1}) = f_1(t)$

$L(x_{p_2}) = f_2(t)$

$L(x_{p_1} + x_{p_2}) = L(x_{p_1}) + L(x_{p_2}) = f_1(t) + f_2(t)$

$f_1: \quad \alpha = 0 \quad , \beta = 1$

$$\left.\begin{array}{ll} P_n \equiv 1 & , n=0 \\ Q_m \equiv 0 & , m=-\infty \end{array}\right\} k=0 \Rightarrow \underline{R_0 = c_1},\ \underline{T_0 = c_2}$$

$$\alpha \pm i\beta = 0 \pm i\cdot 1 = \pm i \Rightarrow \underline{\underline{s=1}} \qquad x_{p_1}(t) = t\cdot\left(c_1\cdot\cos t + c_2\cdot\sin t\right) \ \vdots \ c_1 = c_2 = -\frac{1}{4}$$

f_2: $\alpha = 1$

$\beta = 0$

$$\left.\begin{array}{l} P_n(t) = 2 \Rightarrow n=0 \\ Q_m(t) = \text{било шта } (m=0) \end{array}\right\} k=0 \quad , \ R_0 = c_1,\ T_0 = c_2$$

$$\alpha \pm i\beta = 1 \pm i\cdot 0 = 1 \Rightarrow s=1 \qquad x_{p_2}(t) = t\cdot e^t\cdot(c_1) \ \vdots \ c_1 = 1$$

в) $x'' - x = \sin^2 t$

$$f(t) = \sin^2 t = \frac{1-\cos 2t}{2} = \underbrace{\frac{1}{2}}_{x_{p_1}} - \underbrace{\frac{\cos 2t}{2}}_{x_{p_2}}$$

г) $x'' - 4x' + 5x = (\sin t + 2\cos t)\cdot e^{2t}$

$2 \pm i$

$\alpha = 2$

$\beta = 1$

$$\left.\begin{array}{l} P_n(t) \equiv 2 \Rightarrow n=0 \\ Q_m(t) \equiv 1 \Rightarrow m=0 \end{array}\right\} k=0 \Rightarrow R_0(t) \equiv c_1,\ T_0(t) \equiv c_2$$

$$\alpha \pm i\beta = \underline{2 \pm i} \Rightarrow s = 1$$

$$\Rightarrow x_p(t) = t\cdot e^{2t}\cdot\left(c_1\cos t + c_2\sin t\right) \ \vdots$$

д) $x'' - 2x' + x = \dfrac{e^t}{t}$ → није у одг. облику

$(\lambda-1)^2 = 0 \rightsquigarrow 1,1 \rightsquigarrow e^t,\ te^t$

идеја: $x_p(t) = e^t\cdot g(t)$

$$x_p'(t) = e^t\cdot g(t) + e^t\cdot g'(t) = e^t\cdot\left(g(t) + g'(t)\right)$$

$$x_p''(t) = e^t\cdot\left(g(t) + g'(t)\right) + e^t\cdot\left(g'(t) + g''(t)\right) = e^t\cdot\left(g(t) + 2g'(t) + g''(t)\right)$$

$$x_p'' - 2x_p' + x_p = \frac{e^t}{t}$$

$$e^t \cdot (g + 2g' + g'') - 2e^t (g + g') + e^t g = \frac{e^t}{t} \quad / : e^t$$

$$g'' = \frac{1}{t} \quad / \int \quad \Rightarrow \quad g' = \ln|t| + c_1 \quad \Rightarrow \quad g(t) = t \cdot \ln|t| - t + c_1 t + c_2 = t \cdot \ln|t|$$

$$\left.\begin{matrix} c_1 = 1 \\ c_2 = 0 \end{matrix}\right\}$$

$$x_p(t) = e^t \cdot t \cdot \ln|t|$$

$$\text{OP: } x(t) = c_1 e^t + c_2 e^t \cdot t + e^t \cdot t \cdot \ln|t| \quad , \; c_1, c_2 \in \mathbb{R}$$