① $X' = AX$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix}$$

$\lambda_1 = \lambda_2 = \lambda_3 = 2 \Rightarrow k = 3$

$(A-2E)\gamma = 0 \Rightarrow \gamma = a \cdot \underline{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} + b \cdot \underline{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}} \Rightarrow m = \dim \operatorname{Ker}(A-2E) = 2 \Rightarrow$ 2 блока

$k=3, m=2$: једна могућност $\quad D = \begin{bmatrix} 2 & 1 & \\ & 2 & \\ & & 2 \end{bmatrix}, \quad e^{tD} = \begin{bmatrix} e^{t\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix}} & \\ & e^{2t} \end{bmatrix} = e^{2t} \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

$P = ?$ $\quad \gamma_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \gamma_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ ⇝ треба 1 уопштени ⇝ да ли од γ_1 или γ_2?

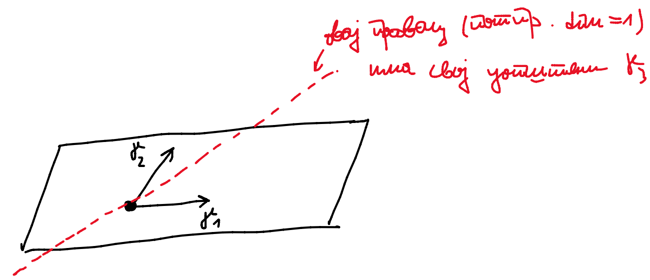
$(A-2E)\gamma_3 = \gamma_1$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} b+c &= 1 \\ -b-c &= 0 \\ b+c &= 0 \end{aligned}\right\} ↯$$

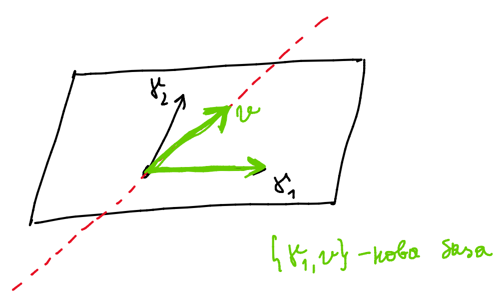
$(A-2E)\gamma_3 = \gamma_2$

$$\left.\begin{aligned} b+c &= 0 \\ -b-c &= 1 \\ b+c &= -1 \end{aligned}\right\} ↯$$



сопствени потпростор:
$\operatorname{Ker}(A-2E) = \mathcal{L}in\{\gamma_1, \gamma_2\}$

изаберемо базу од $\operatorname{Ker}(A-2E)$:



$v = ?$ $\quad v \in \mathcal{L}in\{\gamma_1, \gamma_2\} \Rightarrow v = \alpha \cdot \gamma_1 + \beta \cdot \gamma_2$

→ хоћемо да v има уопштени

$\exists \gamma_3: \quad (A-2E)\gamma_3 = v$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{aligned} b+c &= \alpha \\ -b-c &= \beta \end{aligned} \quad \Rightarrow \quad \alpha = -\beta$$

```
>> A=[2 1 1; 0 1 -1; 0 1 3]

A =
```

$$\begin{aligned} b+c &= \alpha \\ -b-c &= \beta \\ b+c &= -\beta \end{aligned} \quad \Rightarrow \quad \alpha = -\beta$$

```
>> A=[2 1 1; 0 1 -1; 0 1 3]

A =

     2     1     1
     0     1    -1
     0     1     3

>> [P D] = jordan(A)

P =

     1     0     0
    -1     1    -1
     1     0     1


D =

     2     1     0
     0     2     0
     0     0     2
```

nap. $\alpha = 1,\ \beta = -1$

$b+c=1,\ b=1,\ c=0,\ a=0$

$$\Rightarrow \gamma_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\gamma_1 \quad \gamma_2 \quad \rightsquigarrow \quad \gamma_1 \quad \underbrace{v}_{\gamma_3} \quad \Rightarrow P = \begin{bmatrix} v & \gamma_3 & \gamma_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

OP: $X(t) = P \cdot e^{tD} \cdot c, \quad c \in \mathbb{R}^3$

② $X' = AX$

$$A = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \qquad \begin{aligned} \lambda_1 = \lambda_2 &= 2 \\ \lambda_{3/4} &= 1 \pm i \end{aligned}$$

$\underline{\lambda_1 = \lambda_2 = 2}$:

$(k=2)$

$$(A-2E)\gamma_1 = 0 \Rightarrow \gamma_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow m=1 \Rightarrow \begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix}$$

$$(A-2E)\gamma_2 = \gamma_1 \Rightarrow \gamma_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$\underline{\lambda_{3/4} = 1 \pm i}$:

$$\underline{1+i} \rightsquigarrow \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$\alpha = \beta = 1$

$$(A-(1+i)E)\gamma_3 = 0 \Rightarrow \gamma_3 = \begin{bmatrix} 1 \\ -2 \\ -1+i \\ -2i \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -2 \\ -1 \\ 0 \end{bmatrix}}_{\operatorname{Re}\gamma_3} + i \cdot \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}}_{\operatorname{Im}\gamma_3}$$

$$D = \begin{bmatrix} \boxed{\begin{matrix} 2 & 1 \\ & 2 \end{matrix}} & \\ & \boxed{\begin{matrix} 1 & 1 \\ -1 & 1 \end{matrix}} \end{bmatrix} \rightsquigarrow e^{tD} = \begin{bmatrix} e^{t\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix}} & \\ & e^{t\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}} \end{bmatrix} = \begin{bmatrix} e^{2t}\begin{bmatrix} 1 & t \\ & 1 \end{bmatrix} & \\ & e^t \cdot R_t \end{bmatrix} = \begin{bmatrix} e^{2t} & t \cdot e^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 & 0 \\ 0 & 0 & e^t \cos t & e^t \sin t \\ 0 & 0 & -e^t \sin t & e^t \cos t \end{bmatrix}.$$

$$P = \begin{bmatrix} \gamma_1 & \gamma_2 & \operatorname{Re}\gamma_3 & \operatorname{Im}\gamma_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

OP: $X(t) = P \cdot e^{tD} \cdot c, \quad c \in \mathbb{R}^4$

```
>> A=[2 0 1 0; 0 1 0 1; 0 0  2 1; 0 -1 0 1]

A =

     2     0     1     0
     0     1     0     1
     0     0     2     1
     0    -1     0     1

>> [P D] = jordan(A)

P =

   0.0000 - 0.2500i   0.0000 + 0.2500i   0.5000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.5000i   0.0000 - 0.5000i   0.0000 + 0.0000i   0.0000 + 0.0000i
  -0.2500 + 0.2500i  -0.2500 - 0.2500i   0.0000 + 0.0000i   0.5000 + 0.0000i
```

```
>> [P D] = jordan(A)

P =

   0.0000 - 0.2500i   0.0000 + 0.2500i   0.5000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.5000i   0.0000 - 0.5000i   0.0000 + 0.0000i   0.0000 + 0.0000i
  -0.2500 + 0.2500i  -0.2500 - 0.2500i   0.0000 + 0.0000i   0.5000 + 0.0000i
   0.5000 + 0.0000i   0.5000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i


D =

   1.0000 - 1.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   1.0000 + 1.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   2.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   2.0000 + 0.0000i
```

③ $X' = AX$

$$A = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \qquad \lambda_{1/2} = \lambda_{3/4} = \pm i$$

$\lambda_1 = i,\ k=2 \qquad \lambda_1 = 0 + 1\cdot i \rightsquigarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\alpha = 0,\ \beta = 1$

$$D = \begin{bmatrix} \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} \\ & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} \end{bmatrix} \quad m=2 \qquad \vee \qquad D = \boxed{\begin{bmatrix} 0 & 1 & \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} \\ -1 & 0 & \\ & & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{bmatrix}} \quad m=1$$

↳ 2 Жорд. блока ↳ 1 Жорд. блок

$(A - iE)\gamma_1 = 0$

$$\gamma_1 = \begin{bmatrix} 0 \\ -i \\ 0 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{Re\,\gamma_1} + i \cdot \underbrace{\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}}_{Im\,\gamma_1} \Rightarrow m = \dim_{\mathbb{C}} Ker(A - iE) = \dim_{\mathbb{C}} Lin\{\gamma_1\} = 1 \Rightarrow 1 \text{ Жорд. бл.}$$

2 реални век

→ 1 комплексни

$$\Rightarrow D = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

ако бихме добили 2 собств. век $\gamma_1, \gamma_2 \in \mathbb{C}^4$

$m=2 \Rightarrow D = \begin{bmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} Re\,\gamma_1 & Im\,\gamma_1 & Re\,\gamma_2 & Im\,\gamma_2 \end{bmatrix}, \quad e^{tD} = \begin{bmatrix} R_t & \\ & R_t \end{bmatrix}.$

$P = ?$ γ_2 – уточнители за γ_1

$$(A - iE)\gamma_2 = \gamma_1 \ \therefore\ \gamma_2 = \begin{bmatrix} 1 \\ 0 \\ -1-i \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + i \cdot \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$P=\begin{bmatrix} \operatorname{Re} v_1 & \operatorname{Im} v_1 & \operatorname{Re} v_2 & \operatorname{Im} v_2 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$D=\begin{bmatrix} R & E \\ O & R \end{bmatrix} \rightsquigarrow e^{tD}=?$$

$$D=\underbrace{\begin{bmatrix} R & \\ & R \end{bmatrix}}_{\dot{J}}+\underbrace{\begin{bmatrix} & E \\ & \end{bmatrix}}_{N}$$

$$R=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$O=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$E=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$(J \cancel{\times} E)$

$$\dot{J}\cdot N=\begin{bmatrix} R & O \\ O & R \end{bmatrix}\begin{bmatrix} O & E \\ O & O \end{bmatrix}=\begin{bmatrix} O & RE \\ O & O \end{bmatrix}=\begin{bmatrix} O & R \\ O & O \end{bmatrix}$$

$$N\cdot\dot{J}=\begin{bmatrix} O & E \\ O & O \end{bmatrix}\begin{bmatrix} R & O \\ O & R \end{bmatrix}=\begin{bmatrix} O & ER \\ O & O \end{bmatrix}=\begin{bmatrix} O & R \\ O & O \end{bmatrix}$$

$$\Rightarrow \dot{J}N=N\dot{J}$$

$$\Downarrow$$

$$e^{tD}\overset{(2)}{=}e^{t\dot{J}}\cdot e^{tN}$$

$$e^{t\dot{J}}=\begin{bmatrix} e^{tR} & \\ & e^{tR} \end{bmatrix}=\begin{bmatrix} R_t & \\ & R_t \end{bmatrix}$$

$$e^{tN}=\overset{4\times 4}{E}+t\cdot N=\begin{bmatrix} E & t\cdot E \\ O & E \end{bmatrix}$$

$$N^2=\begin{bmatrix} O & E \\ O & O \end{bmatrix}\begin{bmatrix} O & E \\ O & O \end{bmatrix}=\begin{bmatrix} O & O \\ O & O \end{bmatrix}$$

$$N^k=O,\ k\geq 2$$

$$\Rightarrow e^{tD}=\begin{bmatrix} R_t & \\ & R_t \end{bmatrix}\cdot\begin{bmatrix} E & tE \\ & E \end{bmatrix}=\begin{bmatrix} R_tE & tR_tE \\ O & R_tE \end{bmatrix}=\begin{bmatrix} R_t & tR_t \\ O & R_t \end{bmatrix}=$$

$$=\begin{bmatrix} \cos t & \sin t & t\cos t & t\sin t \\ -\sin t & \cos t & -t\sin t & t\cos t \\ 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \end{bmatrix}.$$

ОР: $X(t)=Pe^{tD}\cdot c,\ c\in\mathbb{R}^4$

```
>> A=[-1 0 -1 0; 1 0 1 1; 2 0 1 0; 1 -1 0 0]

A =

    -1     0    -1     0
     1     0     1     1
     2     0     1     0
     1    -1     0     0

>> [P D]=jordan(A)

P =

   0.0000 + 0.0000i   0.5000 - 0.5000i   0.0000 + 0.0000i   0.5000 + 0.5000i
   0.5000 + 0.5000i   0.0000 + 0.0000i   0.5000 - 0.5000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 1.0000i   0.0000 + 0.0000i   0.0000 - 1.0000i
   0.5000 - 0.5000i   0.0000 + 0.0000i   0.5000 + 0.5000i   0.0000 + 0.0000i


D =

   0.0000 - 1.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 - 1.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 1.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 1.0000i
```

(4) Методом елиминације решити систем:

$$x_1'=1-\frac{1}{x_2}$$

$$x_2'=\frac{1}{x_1-t}$$

$x_1(t), x_2(t)$

2 јне 1. реда → 1 јне 2. реда

изразите из једне и убаците у другу

$$x_2'=\frac{1}{x_1-t} \Rightarrow \frac{1}{x_2'}=x_1-t \Rightarrow x_1 \overset{(*)}{=} t+\frac{1}{x_2'} \quad /'$$

изражавамо

$$x_1'=1-\frac{1}{(x_2')^2}\cdot x_2''=1-\frac{x_2''}{(x_2')^2}$$

' - зритовано

$$x_1' = 1 - \frac{1}{(x_2')^2} \cdot x_2'' = 1 - \frac{x_2''}{(x_2')^2}$$

$$(1) \Rightarrow \quad 1 - \frac{x_2''}{(x_2')^2} = 1 - \frac{1}{x_2}$$

$$\Downarrow$$

$$\frac{x_2''}{(x_2')^2} = \frac{1}{x_2}$$

$$x_2'' \cdot x_2 - (x_2')^2 = 0 \quad / : x_2^2$$

$$\frac{x_2'' \cdot x_2 - (x_2')^2}{x_2^2} = 0$$

$$(x x')' = x' \cdot x' + x \cdot x''$$

$$\left(\frac{x'}{x}\right)' = \frac{x'' x - (x')^2}{x^2}$$

$$\left(\frac{x_2'}{x_2}\right)' = 0 \quad \Rightarrow \quad \frac{x_2'}{x_2} = c \quad \Rightarrow \quad x_2' = c \cdot x_2 \quad (\text{РП})$$

$$\vdots$$

$$\Rightarrow x_2(t) = c_1 \cdot e^{ct}$$

$$x_1' = 1 - \frac{1}{c_1} e^{-ct} \quad / \int$$

$$x_1 = t + \frac{1}{c c_1} e^{-ct} + c_2 \quad \rightarrow \text{три константе}$$

превише

$$\circledast \Rightarrow \quad x_1 = t + \frac{1}{x_2'} = t + \frac{1}{c c_1 e^{ct}} = t + \frac{1}{c c_1} e^{-ct}$$