8 - четвртак 02.12.2021.

① $X' = AX$

$$A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 1 \end{bmatrix} \qquad \lambda_1 = -3, \quad \lambda_{2/3} = 2 \pm i$$

```
>> A=[-3 0 0; 0 3 -2; 0 1 1]

A =

    -3     0     0
     0     3    -2
     0     1     1

>> eig(A)

ans =

   2.0000 + 1.0000i
   2.0000 - 1.0000i
  -3.0000 + 0.0000i
```

$$\alpha + i\beta \longleftrightarrow \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

$\underline{\underline{2+i}}$

$$D = \begin{bmatrix} \underbrace{\boxed{-3}}_{1\times 1,\ \lambda_1} & \\ & \underbrace{\boxed{\begin{matrix} 2 & 1 \\ -1 & 2 \end{matrix}}}_{2\times 2,\ \lambda_{2/3}} \end{bmatrix} \rightsquigarrow e^{tD} = ?$$

$$B = \begin{bmatrix} \underbrace{B_1}_{k\times k} & O \\ O & \underbrace{B_2}_{(n-k)\times(n-k)} \end{bmatrix} \Rightarrow B^k = \begin{bmatrix} B_1^k & 0 \\ 0 & B_2^k \end{bmatrix} \Rightarrow e^{tB} = \begin{bmatrix} \underbrace{e^{tB_1}}_{k\times k} & O \\ O & \underbrace{e^{tB_2}}_{(n-k)\times(n-k)} \end{bmatrix}$$

↳ блок-дијагонална

$$e^{tD} = \begin{bmatrix} \boxed{e^{t[-3]}} & \\ & \boxed{e^{t\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}}} \end{bmatrix} = \begin{bmatrix} \boxed{e^{-3t}} & & \\ & \boxed{\begin{matrix} e^{2t}\cos t & e^{2t}\sin t \\ -e^{2t}\sin t & e^{2t}\cos t \end{matrix}} \end{bmatrix}$$

подсетник: $e^{t\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}} = e^{t\alpha} \cdot R_{t\beta}$

$\lambda_1 = -3: \quad (A - \lambda_1 E)\gamma_1 = \vec{0} \Rightarrow \gamma_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$

$\underset{\sim\sim}{\lambda_2 = 2+i}: \quad (A - \lambda_2 E)\gamma_2 = \vec{0}$

$$\begin{bmatrix} -3-(2+i) & 0 & 0 \\ 0 & 3-(2+i) & -2 \\ 0 & 1 & 1-(2+i) \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad a, b, c \in \mathbb{C}, \ \gamma_2 \in \mathbb{C}^3$$

$(-5-i)a = 0 \Rightarrow a = 0$

$(1-i)b - 2c = 0 \longrightarrow /\cdot \frac{1+i}{2} \Rightarrow \frac{(1-i)(1+i)}{2} b - 2 \cdot \frac{1+i}{2} c = 0 \Rightarrow \underline{b - (1+i)c = 0}$

$b + (-1-i)c = 0$

$b = (1+i)c$, $c = 1$, $b = 1+i$

$$\gamma_2 = \begin{bmatrix} 0 \\ 1+i \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{Re\,\gamma_2} + i \cdot \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{Im\,\gamma_2}$$

$$\begin{bmatrix} -1 & & \\ & 2 & 1 \\ & -1 & 2 \end{bmatrix} \qquad \lambda_1, \quad \lambda_{2/3}$$

$$P = \begin{bmatrix} \gamma_1 \downarrow & \overbrace{Re\,\gamma_2 \downarrow \quad Im\,\gamma_2 \downarrow} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

```
>> [P D] = eig(A)

P =

   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i
   0.8165 + 0.0000i   0.8165 + 0.0000i   0.0000 + 0.0000i
   0.4082 - 0.4082i   0.4082 + 0.4082i   0.0000 + 0.0000i


D =

   2.0000 + 1.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   2.0000 - 1.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i  -3.0000 + 0.0000i
```

$c = 1 - i$; γ_2, $\gamma_3 = \overline{\gamma_2}$, γ_1; λ_2, λ_3, λ_1

(γ_3 не треба, $\lambda_3 = \overline{\lambda_2} \Rightarrow \gamma_3 = \overline{\gamma_2}$)

ОР: $X(t) = P \cdot e^{tD} \cdot c$, $c \in \mathbb{R}^3$

Жорданова нормална форма: $D = \begin{bmatrix} \boxed{B_1} & & \\ & \boxed{B_2} & \\ & & \ddots & \\ & & & \boxed{B_k} \end{bmatrix}$

$$B_i = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & 1 \\ & & & \ddots & 1 \\ & & & & \lambda \end{bmatrix} \quad \vee \quad B_i = \begin{bmatrix} R & E & & \\ & R & E & \\ & & \ddots & E \\ & & & R \end{bmatrix}$$

$\lambda \in \mathbb{R}$ понавља

$\alpha + i\beta \in \mathbb{C}$ понавља

$R = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$, $E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

пр. $n = 4$, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda$

$A = P \cdot D \cdot P^{-1}$

D може бити:
$$\begin{bmatrix} \boxed{\lambda} & & & \\ & \boxed{\lambda} & & \\ & & \boxed{\lambda} & \\ & & & \boxed{\lambda} \end{bmatrix}, \quad \begin{bmatrix} \boxed{\begin{matrix} \lambda & 1 \\ & \lambda \end{matrix}} & & \\ & \boxed{\lambda} & \\ & & \boxed{\lambda} \end{bmatrix}, \quad \begin{bmatrix} \boxed{\begin{matrix} \lambda & 1 \\ & \lambda \end{matrix}} & \\ & \boxed{\begin{matrix} \lambda & 1 \\ & \lambda \end{matrix}} \end{bmatrix}, \quad \begin{bmatrix} \boxed{\begin{matrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{matrix}} & \\ & \boxed{\lambda} \end{bmatrix}, \quad \begin{bmatrix} \boxed{\begin{matrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & 1 \\ & & & \lambda \end{matrix}} \end{bmatrix}$$

(2) $X' = AX$

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 2$

$k = 4$ – алгебарска вишеструкост

```
>> A=[2 0 0 0; 0 2 1 0; 0 0 2 0; 1 0 0 2]

A =

     2     0     0     0
     0     2     1     0
     0     0     2     0
     1     0     0     2

>> [P D] = eig(A)

P =
```

$(A - 2E)\gamma = 0$

```
>> [P D] = eig(A)

P =

         0    0.0000         0         0
         0         0    1.0000   -1.0000
         0         0         0    0.0000
    1.0000   -1.0000         0         0


D =

     2     0     0     0
     0     2     0     0
     0     0     2     0
     0     0     0     2
```

$(A-2E)v=0$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \vec{0}$$

$$\begin{matrix} 0=0 \\ c=0 \\ 0=0 \\ a=0 \end{matrix} \Rightarrow v = \begin{bmatrix} 0 \\ b \\ 0 \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\cdot b + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\cdot d$$

$\dim(\mathrm{Ker}(A-2E)) = 2 \Rightarrow m=2$ – геометријска вишеструкост

→ имамо 2 Жорданова блока

$$\Rightarrow D = \begin{bmatrix} 2 & 1 & & \\ & 2 & & \\ & & 2 & 1 \\ & & & 2 \end{bmatrix} \quad \vee \quad D = \begin{bmatrix} 2 & 1 & & \\ & 2 & 1 & \\ & & 2 & \\ & & & 2 \end{bmatrix}$$

минимални полином?

$\mu(\lambda)$

$$\varphi(\lambda) = \det(A-\lambda E) = (\lambda-2)^4$$

$$\mu \mid \varphi \Rightarrow \mu(\lambda) = (\lambda-2)^{\ell}, \ \ell \in \{1,2,3,4\}$$

$\mu(A)=0$

$$(A-2E)^1 = \begin{bmatrix} & & & \\ & & 1 & \\ & & & \\ 1 & & & \end{bmatrix} \neq 0$$

$$(A-2E)^2 = 0 \Rightarrow \ell=2 \Rightarrow \mu(\lambda)=(\lambda-2)^{2} \Rightarrow \deg\mu = 2$$

$\Downarrow$

2 је ред највећег Ж. блока

$$\Rightarrow D = \begin{bmatrix} \boxed{\begin{matrix} 2 & 1 \\ & 2 \end{matrix}} & \\ & \boxed{\begin{matrix} 2 & 1 \\ & 2 \end{matrix}} \end{bmatrix}$$

$v_1 \quad v_3 \qquad v_2 \quad v_4$

$P=?$ сопс. век.: $v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$

$v_1 \rightsquigarrow v_3$

$v_2 \rightsquigarrow v_4$

→ уопштени сопс. век. (генерализовани)

$v_1 \rightsquigarrow v_3$: $(A-2E)v_3 = v_1$

$$\begin{matrix} 0=0 \\ c=1 \\ 0=0 \end{matrix} \Rightarrow v_3 = \begin{bmatrix} 0 \\ b \\ 1 \\ d \end{bmatrix} \xrightarrow{b=d=0} v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} 0 &= 0 \\ c &= 1 \\ 0 &= 0 \\ a &= 0 \end{aligned} \Rightarrow v_3 = \begin{bmatrix} 0 \\ b \\ 1 \\ d \end{bmatrix} \xrightarrow{b=d=0} v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$v_2 \rightsquigarrow v_4$: $\quad (A-2E)v_4 = v_2$

$$\begin{aligned} 0 &= b \\ c &= 0 \\ 0 &= 0 \\ a &= 1 \end{aligned} \Rightarrow v_4 = \begin{bmatrix} 1 \\ b \\ 0 \\ d \end{bmatrix} \xrightarrow{b=d=0} v_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$P = \begin{bmatrix} v_1 & v_3 & v_2 & v_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad (v_1 v_3 v_2 v_4 \text{ nebo } v_2 v_4 v_1 v_3)$$

```
>> jordan(A)

ans =

     2     1     0     0
     0     2     0     0
     0     0     2     1
     0     0     0     2

>> [P D]=jordan(A)

P =

     0     1     0     0
     0     0     1     0
     0     0     0     1
     1     0     0     0


D =

     2     1     0     0
     0     2     0     0
     0     0     2     1
     0     0     0     2
```

$e^{tD} = ?$

$$e^{tD} = \begin{bmatrix} e^{t\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix}} & \\ & e^{t\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix}} \end{bmatrix} = e^{2t} \cdot \begin{bmatrix} 1 & t & & \\ & 1 & & \\ & & 1 & t \\ & & & 1 \end{bmatrix}$$

$$e^{t\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix}} = e^{t\begin{bmatrix} 2 & \\ & 2 \end{bmatrix} + t\begin{bmatrix} & 1 \\ & \end{bmatrix}} \overset{(2)}{=} e^{t\begin{bmatrix} 2 & \\ & 2 \end{bmatrix}} \cdot e^{t\overset{N}{\begin{bmatrix} & 1 \\ & \end{bmatrix}}} = \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix} \cdot \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix} = e^{2t}\begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}$$

$$\left.\begin{aligned} t\begin{bmatrix} 2 & \\ & 2 \end{bmatrix} \cdot t\begin{bmatrix} & 1 \\ & \end{bmatrix} &= t^2 \cdot \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \\ t \cdot \begin{bmatrix} & 1 \\ & \end{bmatrix} \cdot t\begin{bmatrix} 2 & \\ & 2 \end{bmatrix} &= t^2 \cdot \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \end{aligned}\right\} \text{komutují}$$

$$N^2 = \begin{bmatrix} & 1 \\ & \end{bmatrix}\begin{bmatrix} & 1 \\ & \end{bmatrix} = 0 \Rightarrow N^k = 0,\ k \geq 2$$

$$e^{tN} = \underset{k=0}{E} + \underset{k=1}{t \cdot N} = \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}$$

OR: $X(t) = P \cdot e^{tD} \cdot c,\ c \in \mathbb{R}^4$

3 $\quad X' = AX$

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix} \qquad \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 2, \quad k = 4$$

$$(A-2E)v = \vec{0} \Rightarrow v \in \text{Lin}\left\{ \overset{v_1}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}, \overset{v_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}} \right\} \Rightarrow m = 2 \Rightarrow 2 \text{ Jb.}$$

$$\begin{bmatrix} 2 & 1 & & \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & & \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & & \\ & 2 & & \\ & & 2 & 1 \\ & & & 2 \end{bmatrix} \vee \begin{bmatrix} 2 & 1 & & \\ & 2 & 1 & \\ & & 2 & \\ & & & 2 \end{bmatrix}$$

$\varphi(\lambda) = (\lambda-2)^4$

$\mu(\lambda) = (\lambda-2)^\ell$

$(A-2E)^1 \neq 0$, $(A-2E)^2 \neq 0$, $(A-2E)^3 = 0 \Rightarrow$ ред највећег блока је 3 $\Rightarrow J = \begin{bmatrix} 2 & 1 & & \\ & 2 & 1 & \\ & & 2 & \\ & & & \boxed{2} \end{bmatrix}$

уопштени за γ_2: $(A-2E)\gamma_3 = \gamma_2$

$$\begin{aligned} 0 &= 0 \\ 0 &= 1 \quad ↯ \\ d &= 0 \\ a &= 0 \end{aligned}$$

$\Rightarrow$ нема реш. $\Rightarrow \gamma_2$ нема уопштени $\xrightarrow{\gamma_2}$

$\Rightarrow \gamma_2$ одговара 1×1

$\gamma_1 \rightsquigarrow \gamma_3 \rightsquigarrow \gamma_4$ $\quad \gamma_2$

уопштени за γ_1: $(A-2E)\gamma_3 = \gamma_1$

$$\begin{aligned} 0 &= 0 \\ 0 &= 0 \\ d &= 1 \\ a &= 0 \end{aligned} \Rightarrow \gamma_3 = \begin{bmatrix} 0 \\ b \\ c \\ 1 \end{bmatrix} \xrightarrow{b=c=0} \gamma_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

γ_3: $(A-2E)\gamma_4 = \gamma_3$

$$\begin{aligned} 0 &= 0 \\ 0 &= 0 \\ d &= 0 \\ a &= 1 \end{aligned} \Rightarrow \gamma_4 = \begin{bmatrix} 1 \\ b \\ c \\ 0 \end{bmatrix} \xrightarrow{b=c=0} \gamma_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$J = \begin{bmatrix} \boxed{} & \\ & \square \end{bmatrix} \rightsquigarrow P = \begin{bmatrix} \gamma_1 & \gamma_3 & \gamma_4 & \gamma_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

$$e^{tJ} = \begin{bmatrix} e^{t\begin{bmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{bmatrix}} & \\ & e^{t[2]} \end{bmatrix} = \begin{bmatrix} e^{2t}\begin{bmatrix} 1 & t & \frac{t^2}{2} \\ & 1 & t \\ & & 1 \end{bmatrix} & \\ & e^{2t} \end{bmatrix} = e^{2t} \cdot \begin{bmatrix} 1 & t & \frac{t^2}{2} & \\ & 1 & t & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

$2E$ $\quad N$ ← увек нилпотентна

увек нилпотентна

$$B=\begin{bmatrix}2&1&\\&2&1\\&&2\end{bmatrix},\quad e^{tB}=e^{t\overset{=2E}{\begin{bmatrix}2&&\\&2&\\&&2\end{bmatrix}}+t\overset{=N}{\begin{bmatrix}&1&\\&&1\\&&\end{bmatrix}}}\overset{(2)}{=}e^{t\begin{bmatrix}2&&\\&2&\\&&2\end{bmatrix}}\cdot e^{tN}=e^{2t}\cdot E\cdot\begin{bmatrix}1&t&\frac{t^2}{2}\\&1&t\\&&1\end{bmatrix}=e^{2t}\begin{bmatrix}1&t&\frac{t^2}{2}\\&1&t\\&&1\end{bmatrix}.$$

$2E\cdot N=N\cdot 2E$

$$N^2=\begin{bmatrix}&1&\\&&1\\&&\end{bmatrix}\begin{bmatrix}&1&\\&&1\\&&\end{bmatrix}=\begin{bmatrix}&&1\\&&\\&&\end{bmatrix}$$

$$N^3=O\ ,\quad N^k=O,\ k\geq 3$$

$$e^{tN}=\underset{k=0}{\underset{\uparrow}{E}}+\underset{k=1}{\underset{\uparrow}{t\cdot N}}+\underset{k=2}{\underset{\uparrow}{\frac{t^2}{2}\cdot N^2}}=\begin{bmatrix}1&t&\frac{t^2}{2}\\&1&t\\&&1\end{bmatrix}$$

OP: $X(t)=P\cdot e^{tD}\cdot c,\ c\in\mathbb{R}^4$

```
>> A=[2 0 0 0; 0 2 0 0; 1 0 2 1; 1 0 0 2]

A =

     2     0     0     0
     0     2     0     0
     1     0     2     1
     1     0     0     2

>> [P D]=jordan(A)

P =

     0     0     1     0
     0     0     0     1
     1     1     0     0
     0     1     0     0


D =

     2     1     0     0
     0     2     1     0
     0     0     2     0
     0     0     0     2
```