$\boxed{\textbf{Тврђење 52.}}$ **(Својства експонента.)**

(1) $e^{\mathbf{0}} = \mathrm{Id}$; 0-нула матрица, $Id = I = E = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$

(2) $AB = BA \Rightarrow e^{A+B} = e^A e^B$;

(3) $AB = BA \Rightarrow Be^A = e^A B$; специ: $A = B$, $Ae^A = e^A A$

(4) $e^A = \lim\limits_{n\to\infty} (\mathrm{Id} + \frac{A}{n})^n$;

(5) за $U = \mathbb{R}^n$, тј. $A \in M_n(\mathbb{R})$ важи $\frac{d}{dt}e^{tA} = e^{tA}A = Ae^{tA}$;

(6) за $U = \mathbb{R}^n$ важи $\det e^A = e^{\operatorname{tr} A}$;

(7) за $U = \mathbb{R}^n$ важи $e^{P^{-1}AP} = P^{-1}e^A P$.

Домаћи: (2) не важи за $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

в) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\quad (\lambda_{1/2} = \pm i)$

$e^{tA} = ?$ $\quad A^k = ?$

$$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -E$$

$$A^3 = A^2 A = -E \cdot A = -A$$

$$A^4 = A^3 \cdot A = (-A)\cdot A = -A^2 = -(-E) = E = A^0$$

$$A^k = A^{4t+r} = (A^4)^t \cdot A^r = E^t \cdot A^r = A^r = \begin{cases} E, & r=0 \\ A, & r=1 \\ -E, & r=2 \\ -A, & r=3 \end{cases} = \begin{cases} (-1)^\ell E, & k = 2\ell \\ (-1)^\ell A, & k = 2\ell+1 \end{cases}$$

$k = 4t + r$

$r \in \{0,1,2,3\}$

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = \sum_{\ell=0}^{\infty} \frac{t^{2\ell} \cdot A^{2\ell}}{(2\ell)!} + \sum_{\ell=0}^{\infty} \frac{t^{2\ell+1} \cdot A^{2\ell+1}}{(2\ell+1)!} = \underbrace{\sum_{\ell=0}^{\infty} \frac{t^{2\ell}(-1)^\ell}{(2\ell)!}}_{1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots = \cos t} \cdot E + \underbrace{\sum_{\ell=0}^{\infty} \frac{t^{2\ell+1}\cdot(-1)^\ell}{(2\ell+1)!}}_{t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots = \sin t} \cdot A = \cos t \cdot E + \sin t \cdot A$$

$$= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} = R_t$$

ОР: $X(t) = R_t \cdot c$, $c \in \mathbb{R}^2$

г) $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

представља: $A^k \leftrightarrow (a+ib)^k$

II начин: $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = aE + bB$, $\quad B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$e^{tA} = e^{t(aE+bB)} = e^{taE + tbB} \overset{(2)}{=} e^{taE} \cdot e^{tbB} = \underbrace{\dots}_{\text{домаћи}} = e^{ta} \cdot R_{tb}.$$

$(taE)\cdot(tbB) = t^2ab\,\overbrace{E \cdot B}^{B} = t^2ab\,BE = (tbB)(taE) \Rightarrow$ комутир.

① Испитати да ли $\exists A\in M_2(\mathbb{R})$ тд:

а) $e^A=\begin{bmatrix}1 & 0\\ 0 & -4\end{bmatrix}$ б) $e^A=\begin{bmatrix}-1 & 0\\ 0 & -4\end{bmatrix}$

у $\mathbb{R}$:
$e^a=b$
$b>0$ ✓
$b\le 0$ ×

а) (6) $\Rightarrow$ $\det(e^A)=e^{\overset{\in\mathbb{R}}{trA}}>0$

$$\left.\begin{array}{l}\det(e^A)=e^{trA}>0\\ \det\left(\begin{bmatrix}1 & 0\\ 0 & -4\end{bmatrix}\right)=-4<0\end{array}\right\}\ \lightning$$

б) $\det\left(\begin{bmatrix}-1 & 0\\ 0 & -4\end{bmatrix}\right)=4>0$

$4=e^{trA}\Rightarrow trA=\ln 4$

(3) $\Rightarrow$ $Ae^A=e^AA$ $\qquad A=\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}$

$$\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\begin{bmatrix}-1 & \\ & -4\end{bmatrix}=\begin{bmatrix}-1 & \\ & -4\end{bmatrix}\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}$$

$-\alpha=-\alpha$ ×
$\left.\begin{array}{l}-4\beta=-\beta\\ -\gamma=-4\gamma\end{array}\right\}\ \beta=\gamma=0\ \Rightarrow\ A=\begin{bmatrix}\alpha & 0\\ 0 & \delta\end{bmatrix}$
$-4\delta=-4\delta$ ×

(*) $A=\mathrm{diag}\{\lambda_1,\dots,\lambda_n\}\Rightarrow e^A=\mathrm{diag}\{e^{\lambda_1},\dots,e^{\lambda_n}\}$

след: $e^A=\begin{bmatrix}e^\alpha & 0\\ 0 & e^\delta\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & -4\end{bmatrix}\Rightarrow\left.\begin{array}{l}e^\alpha=-1\\ e^\delta=-4\end{array}\right\}\ \lightning$

② $\lambda\in\mathbb{C}$ сопс. вр. A $\Rightarrow e^\lambda$ сопс. вр. e^A

I начин: $\exists v\neq 0:\ \lambda v=Av$

$?\exists v_1\neq 0:\ e^\lambda v_1=e^Av_1$ ⇝ узећемо баш $v_1=v$

$$e^Av=\left[\sum_{k=0}^{\infty}\frac{A^k}{k!}\right]\cdot v=\sum_{k=0}^{\infty}\frac{A^kv}{k!}=\sum_{k=0}^{\infty}\frac{\lambda^kv}{k!}=\left[\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\right]\cdot v=e^\lambda\cdot v$$

$$\underline{A^kv}=A^{k-1}(Av)=A^{k-1}\lambda v=\lambda\cdot\underline{A^{k-1}v}=\lambda^2A^{k-2}v=\dots=\lambda^kv$$

II начин: $\det(A-\lambda E)=0$

$? \det(e^A - e^\lambda E) = 0$

$$\det(e^A - e^\lambda E) = \det\left(\sum_{k=0}^{\infty} \frac{A^k}{k!} - \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} E\right) = \det\left(\sum_{k=0}^{\infty} \frac{A^k - (\lambda E)^k}{k!}\right) = \dots$$

gotovo

③ $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Naći $\det(e^{e^A})$.

$A, e^A, e^{e^A} \in M_3(\mathbb{R})$

(6) $\Rightarrow \det(e^{e^A}) = e^{tr(e^A)}$

$e^A = ?$

$$A = \underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}}_{E} + \underbrace{\begin{bmatrix} & & 1 \\ & & \\ 1 & & \end{bmatrix}}_{B}$$

$EB = B = BE \overset{(2)}{\Rightarrow} e^A = e^{B+E} = e^B \cdot e^E$

$E = diag\{1,1,1\} \Rightarrow e^E = diag\{e,e,e\} = e \cdot E$

$B^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$B^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = B$

$B^4 = (B^2)^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\left.\begin{array}{l} B^0 = E \\ B^1 = B^3 = \dots = B \\ B^2 = B^4 = \dots = B^2 \end{array}\right\} \text{indukcija}$$

$$e^B = \sum_{k=0}^{\infty} \frac{B^k}{k!} = \underset{\substack{\uparrow \\ k=0}}{E} + \sum_{l=1}^{\infty} \frac{B^{2l}}{(2l)!} + \sum_{l=0}^{\infty} \frac{B^{2l+1}}{(2l+1)!} =$$

$$= E + B^2 \cdot \Big(\underbrace{\sum_{l=0}^{\infty} \frac{1}{(2l)!}}_{ch 1} \underset{\substack{\nearrow \\ l=0}}{-1}\Big) + B \cdot \underbrace{\sum_{l=0}^{\infty} \frac{1}{(2l+1)!}}_{sh 1}$$

$= E + B^2(ch 1 - 1) + B \, sh 1 =$

$= \begin{bmatrix} ch 1 & 0 & sh 1 \\ 0 & 1 & 0 \\ sh 1 & 0 & ch 1 \end{bmatrix}$

$$\left.\begin{array}{l} e = \sum_{k=0}^{\infty} \frac{1}{k!} = \Pi + H \\ e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = \Pi - H \end{array}\right\} \quad \Pi = \frac{e + e^{-1}}{2} = ch 1$$

$H = \frac{e - e^{-1}}{2} = sh 1$

$$\Rightarrow e^A = eE\cdot e^B = e\cdot e^B \Rightarrow \operatorname{tr} A = e\cdot \operatorname{tr} B = e\,(\operatorname{ch}1+1+\operatorname{ch}1) = e\,(e+e^{-1}+1) = e^2+e+1$$

$$\det(e^{e^A}) = e^{e^2+e+1}.$$

(4) Нека је $A=[a_{ij}]_{i,j=1}^{n} \in M_n(\mathbb{R})$, $a_{ij}\geq 0, \forall i\neq j$. Нека је $B=e^A=[b_{ij}]_{i,j=1}^{n} \in M_n(\mathbb{R})$.

Доказати $b_{ij}\geq 0$, $\forall i,j$.

$$A = \begin{bmatrix} \ddots & \geq 0 \\ \geq 0 & \ddots \end{bmatrix} \rightsquigarrow B = e^A = \begin{bmatrix} \geq 0 \end{bmatrix}.$$

(*) Ако $a_{ij}\geq 0, \forall i,j \Rightarrow b_{ij}\geq 0, \forall i,j$

$$\text{нпр. } A = \begin{bmatrix} 7 & & \\ & -8 & \\ & & -2 \end{bmatrix} = \underbrace{\begin{bmatrix} -8 & & \\ & -8 & \\ & & -8 \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} 15 & & \\ & 0 & \\ & & 6 \end{bmatrix}}_{Y}, \quad XY=YX \Rightarrow e^A = e^X\cdot e^Y$$

$A = X+Y$

$X = M\cdot E$, $Y = A - ME$

$M = \min\{a_{ii} \mid 1\leq i\leq n\}$

$$Y = \begin{cases} a_{ij} \geq 0, & i\neq j \\ a_{ii} - M \end{cases} = [\geq 0] \Rightarrow e^Y = [\geq 0]$$

≥ 0 (јер је $M = \min$)

$XY = YX$? $\quad X = ME$, $ME\cdot Y = M\cdot Y = MYE = YME = YX$ ✓ $\overset{(2)}{\Rightarrow} e^A = e^X\cdot e^Y$

$X = \operatorname{diag}\{M,\dots,M\} \Rightarrow e^X = \operatorname{diag}\{e^M,\dots,e^M\} = e^M\cdot E$

$$B = e^A = e^M E\cdot e^Y = \underbrace{e^M}_{\geq 0}\cdot \underbrace{e^Y}_{[\geq 0]} = [\geq 0] \Rightarrow \forall i,j: b_{ij}\geq 0.$$

Решавање ЛСДЈКК свођењем на Жорданову форму

$X' = AX$

ОР: $X(t)=e^{tA}\cdot C \;, C\in\mathbb{R}^n$

$$A\sim D \;,\quad A=P\cdot D\cdot P^{-1} \;,\quad P\in GL_n(\mathbb{R})$$

→ матрица преласка

↳ у Жордановој нормалној форми

$$e^{tA}=e^{tPDP^{-1}} \overset{(7)}{=} P\,e^{tD}\,P^{-1}$$

↳ знамо да израчунамо

(5)
$$x_1' = x_1-x_2+x_3$$
$$x_2' = x_1+x_2-x_3$$
$$x_3' = 2x_1-x_2$$

а) Нађи ОР.

б) Нађи ПР $X(0)=\begin{bmatrix}1\\1\\1\end{bmatrix}$.

```
>> A=[1 -1 1; 1 1 -1; 2 -1 0]

A =

     1    -1     1
     1     1    -1
     2    -1     0

>> eig(A)

ans =

   -1.0000
    1.0000
    2.0000
```

а) $X'=AX \;, A=\begin{bmatrix}1&-1&1\\1&1&-1\\2&-1&0\end{bmatrix}$

$$\det(A-\lambda E)=0$$

$$\begin{vmatrix}1-\lambda & -1\\ -1 & -\lambda\end{vmatrix}$$

$$0=\begin{vmatrix}1-\lambda & -1 & 1\\ 1 & 1-\lambda & -1\\ 2 & -1 & -\lambda\end{vmatrix} = (1-\lambda)\cdot\left((1-\lambda)(-\lambda)-(-1)^2\right) - (-1)\cdot\left(-\lambda+2\right) + 1\cdot\left(-1-2(1-\lambda)\right) =$$

$$= (1-\lambda)\left(-\lambda+\lambda^2-1\right) + \underbrace{2-\lambda-1-2+2\lambda}_{\lambda-1} =$$

$$=(1-\lambda)(\lambda^2-\lambda-1-1) = (1-\lambda)(\lambda-2)(\lambda+1)$$

$$\lambda_1=-1,\quad \lambda_2=1,\quad \lambda_3=2 \;\Rightarrow\; D=\begin{bmatrix}-1&&\\&1&\\&&2\end{bmatrix} \;\Rightarrow\; e^{tD}=\begin{bmatrix}e^{-t}&&\\&e^{t}&\\&&e^{2t}\end{bmatrix}$$

$$P=? \;,\quad P=\begin{bmatrix}\gamma_1\downarrow & \gamma_2\downarrow & \gamma_3\downarrow\end{bmatrix}$$

↳ сопс. век.

$\lambda_1=-1$: $(A-\lambda_1 E)\gamma_1=0$

$$\begin{bmatrix}2&-1&1\\1&2&-1\\2&-1&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$2a-b+c=0$$
$$a+2b-c=0$$
$$2a-b+c=0$$

$$\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} \cdot \\ \cdot \\ c \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ 0 \end{bmatrix}$$

$$2a - b + c = 0$$
$$a + 2b - c = 0$$
$$\underline{2a - b + c = 0}$$
$$\vdots$$
$$b = -3a$$
$$c = -5a$$

$$\begin{bmatrix} a \\ -3a \\ -5a \end{bmatrix} = a \cdot \underbrace{\begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix}}_{= v_1} \quad (a=1)$$

$$\lambda_2 = 1: \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \lambda_3: \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

```
>> [P D] = eig(A)

P =
        v1         v2        v3
    0.1690    -0.5774    0.7071
   -0.5071    -0.5774    0.0000
   -0.8452    -0.5774    0.7071

D =

   -1.0000          0         0
         0     1.0000         0
         0          0    2.0000
      λ1         λ2        λ3
```

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 1 & 0 \\ -5 & 1 & 1 \end{bmatrix}$$

```
P =

     1     1     1
    -3     1     0
    -5     1     1

>> inv(P)

ans =

    0.1667    0.0000   -0.1667
    0.5000    1.0000   -0.5000
    0.3333   -1.0000    0.6667
```

$$P^{-1} = ?$$

$$P^{-1} = \frac{1}{\det P} \cdot \operatorname{Adj} P = \frac{1}{6} \cdot \begin{bmatrix} +\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} -3 & 0 \\ -5 & 1 \end{vmatrix} & +\begin{vmatrix} -3 & 1 \\ -5 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ -5 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ -5 & 1 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ -3 & 0 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ -3 & 1 \end{vmatrix} \end{bmatrix}^T = \frac{1}{6}\begin{bmatrix} 1 & 3 & 2 \\ 0 & 6 & -6 \\ -1 & -3 & 4 \end{bmatrix}^T = \frac{1}{6}\begin{bmatrix} 1 & 0 & -1 \\ 3 & 6 & -3 \\ 2 & -6 & 4 \end{bmatrix}$$

$$\text{OP}: X(t) = P \cdot e^{tD} \cdot \underbrace{\boxed{P^{-1} \cdot c}}_{= c_1} = P \cdot e^{tD} \cdot c_1 \qquad c_1, c \in \mathbb{R}^3$$

```
>> A=[1 -1 1; 1 1 -1; 2 -1 0]

A =

     1    -1     1
     1     1    -1
     2    -1     0

>> syms t
>> expm(A*t)

ans =

[     exp(-t)/6 + exp(2*t)/3 + exp(t)/2, exp(t) - exp(2*t),     (2*exp(2*t))/3 - exp(-t)/6 - exp(t)/2]
[                  exp(t)/2 - exp(-t)/2,            exp(t),                      exp(-t)/2 - exp(t)/2]
[ exp(2*t)/3 - (5*exp(-t))/6 + exp(t)/2, exp(t) - exp(2*t), (5*exp(-t))/6 + (2*exp(2*t))/3 - exp(t)/2]
```

б) $X(t) = P \cdot e^{tD} \cdot c_1$

$$X(0) = P \cdot c_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow c_1 = P^{-1} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$X(t) = P \cdot e^{tD} \cdot P^{-1} \cdot c$$

$$X(0) = c = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow X(t) = \underbrace{e^{tA}}_{= Pe^{tD}P^{-1}} \cdot X(0)$$