(*) На предавањима је Пикарова Т формулисана за C^1 фје ⇝ у доказу се користи $\boxed{\text{лок. Лип.}}$

→ предавања: $F(x,t) \rightsquigarrow F(x)$, $\|F(x)-F(y)\| \le L\cdot\|x-y\|$

→ вежбе: $F(x,t)$ $\|F(x,t)-F(y,t)\| \le L\cdot\|x-y\|$

(1) $x'=x(1-x)$. Без решавања:

а) Свако решење тд. $x(0)=a\in(0,1) \Rightarrow (\forall t)\ 0<x(t)<1$.

б) Наћи $\lim\limits_{t\to\infty} x(t)$ у зависности од $x(0)=a\in\mathbb{R}$.

а) Пикар:

$$\left.\begin{aligned} x'&=x(1-x)\\ x(0)&=a\end{aligned}\right\}$$

$F(x,t)=x(1-x)$ је $\underset{\text{(нема } t)}{\text{аутономно}}$ и C^1

$\frac{\partial F}{\partial x}=1-2x \in C(\mathbb{R})$

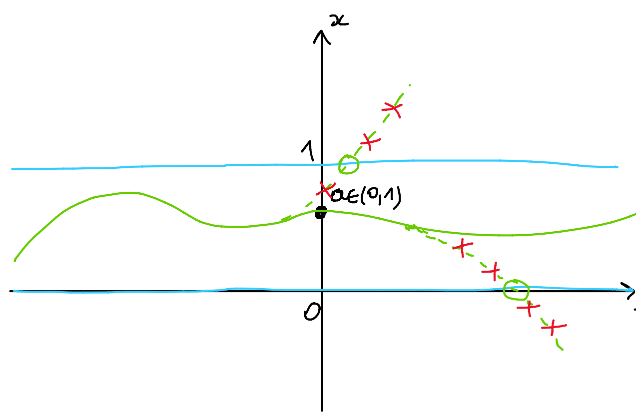
⇒ F је C^1 ⇒ важи Пикар

⇒ јединствено решење

⇒ реш. се не секу

$x\equiv 0$, $0'=0\cdot1=0$ ✓

$x\equiv 1$, $1'=1\cdot0=0$ ✓



$x(0)=a\in(0,1) \Rightarrow x(t)$ унутар траке $\mathbb{R}\times(0,1) \Rightarrow (\forall t)\ 0<x(t)<1$.

б) $x(0)=a\in\mathbb{R}$

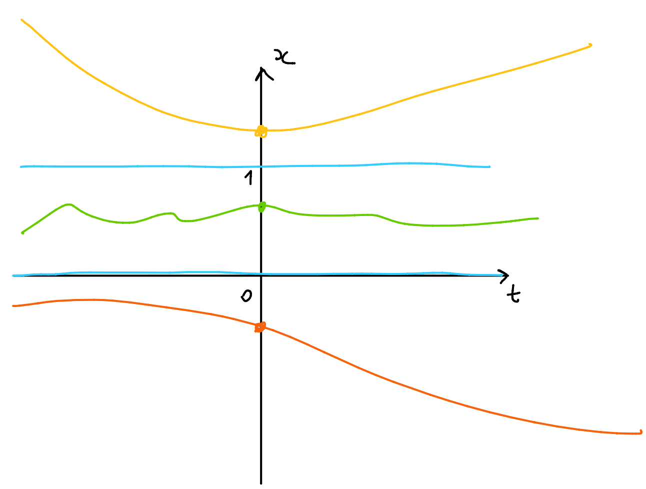
1° $a\in(0,1) \Rightarrow x(t)\in(0,1)$

2° $a=0 \Rightarrow x(t)\equiv0$

3° $a=1 \Rightarrow x(t)\equiv1$

4° $a>1 \Rightarrow x(t)>1$

5° $a<0 \Rightarrow x(t)<0$

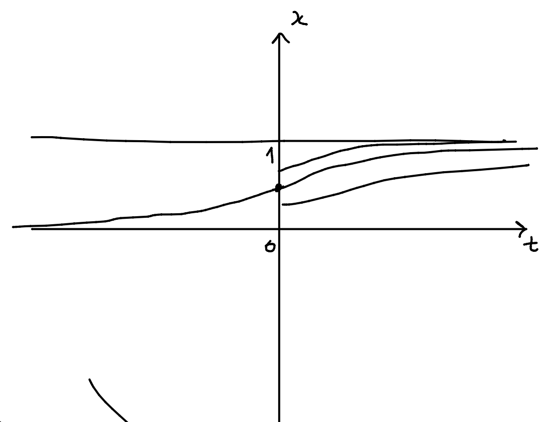


1° $x \in (0,1)$

$$x' = \underbrace{x}_{\in(0,1)}\underbrace{(1-x)}_{\in(0,1)} \in (0,1) \Rightarrow x' > 0 \Rightarrow x \nearrow$$

$x\nearrow$, x огр. са 1, $x \in C^1 \Rightarrow x$ има хор. асимпт. и мора важити

$$x'(t) \xrightarrow[t\to\infty]{} 0$$



$x' \to 0$, $x(1-x) \to 0 \Rightarrow x \to 0 \vee x \to 1$

због $x\nearrow$

$\Rightarrow \lim\limits_{t\to\infty} x(t) = 1.$

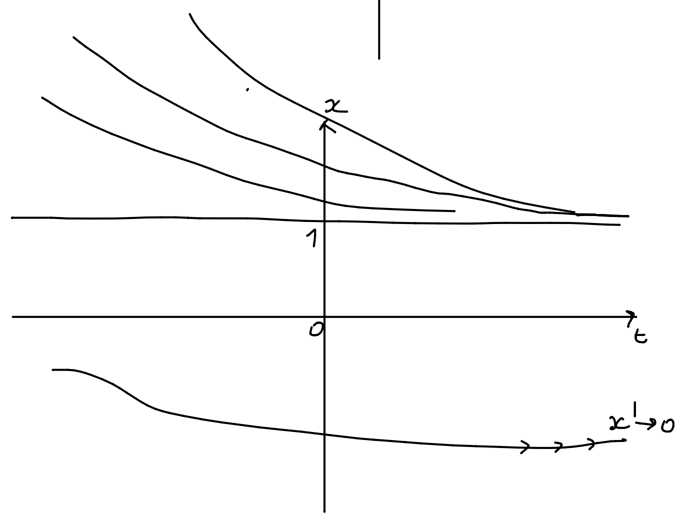
2° $x(t) > 1$

$$x' = \underbrace{x}_{>1}\underbrace{(1-x)}_{<0} \Rightarrow x' < 0 \Rightarrow x \searrow$$

хор. асимпт?

$$\underbrace{x' \to 0}_{x>1} \Rightarrow x \to 1$$

$\Rightarrow \lim\limits_{t\to\infty} x(t) = 1.$



3° $x(t) < 0$

$$x' = \underbrace{x}_{<0}\underbrace{(1-x)}_{>1} < 0 \Rightarrow x' < 0 \Rightarrow x \searrow$$

$\Rightarrow$ не могу имати хор. асимпт. $(x' \to 0 \Rightarrow x \to 0 \vee x \to 1)$ ↯

$$\left.\begin{array}{l} x \to -\infty \\ x' \to -\infty \end{array}\right\} \lim_{t\to\infty} x(t) = -\infty$$

$$\lim_{t\to\infty} x(t) = \begin{cases} 1, & a > 0 \\ 0, & a = 0 \\ -\infty, & a < 0 \end{cases}$$

② Доказати да $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = \left(\sqrt{x^2+y^2}, \sqrt[4]{x^2+y^2}\right)$ није локално Липшицова ни у једној околини $(0,0)$.

тј. $X' = f(X)$, $X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

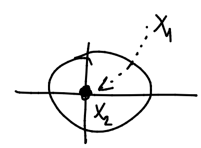
$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} \sqrt{x(t)^2 + y(t)^2} \\ \sqrt[4]{x(t)^2 + y(t)^2} \end{bmatrix}$$

$$\Updownarrow$$

$$x' = \sqrt{x^2+y^2}$$
$$y' = \sqrt[4]{x^2+y^2}$$

не важи Пикар за $X(t_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$\|f(X_1) - f(X_2)\| \le L \cdot \|X_1 - X_2\|$, $X_1 = (x_1, y_1)$, $X_2 = (x_2, y_2)$ у околини $(0,0)$



узмемо $X_2 = (0,0) \longrightarrow f(X_2) = (0,0)$

X_1 у произв. малој околини $(0,0)$, $X_1 \to (0,0)$

$$\|f(X_1)\| \le L \cdot \|X_1\|$$

$\|\cdot\|$ - узмемо еуклидску норму $\|\cdot\|_2$

$$\left\| \left(\sqrt{x_1^2+y_1^2}, \sqrt[4]{x_1^2+y_1^2} \right) \right\|_2 \le L \cdot \|(x_1, y_1)\|_2$$

$$\sqrt{\left(\sqrt{x_1^2+y_1^2}\right)^2 + \left(\sqrt[4]{x_1^2+y_1^2}\right)^2} \le L \cdot \sqrt{x_1^2+y_1^2} \quad /^2$$

$$x_1^2+y_1^2 + \sqrt{x_1^2+y_1^2} \le L^2 \cdot (x_1^2+y_1^2) \quad /: (x_1^2+y_1^2)$$, $X_1 \neq (0,0)$

$$1 + \boxed{\frac{1}{\sqrt{x_1^2+y_1^2}}} \le L^2 \quad / \lim_{X_1 \to (0,0)}$$

$\boxed{\frac{1}{\sqrt{x_1^2+y_1^2}}} \to +\infty$

$$\lim_{(x_1,y_1)\to(0,0)} \frac{1}{\sqrt{x_1^2+y_1^2}} = +\infty$$

$$\infty \le L^2 \in \mathbb{R}$$
$$\Downarrow$$
$$\lightning$$

③ формирати низ итерација из доказа Пикарове Т за проблем: $x' = \frac{x}{t}$, $x(t_0) = x_0$, $t_0 > 0$

$x_0(t) \equiv x_0$

$x_{n+1}(t) = x_0 + \int^t F(x_n(s), s)\, ds$

$F(x,t) = \frac{x}{t}$

t_0

(Пред: $F(x_n(s))$) $x_n \to x_\infty$ ↳ решење ДЈ.

$x_0(t) \equiv x_0$

$$x_1(t) = x_0 + \int_{t_0}^{t} F(\overbrace{x_0(s)}^{x_0}, s)\,ds = x_0 + \int_{t_0}^{t} \frac{x_0}{s}\,ds = x_0 + x_0(\ln s)\Big|_{t_0}^{t} = x_0 + x_0(\ln t - \ln t_0) = x_0 + x_0 \ln\frac{t}{t_0}$$

$$x_2(t) = x_0 + \int_{t_0}^{t} F(x_1(s), s)\,ds = x_0 + \int_{t_0}^{t} \frac{x_0 + x_0\ln\frac{s}{t_0}}{s}\,ds = x_0 + x_0\ln\frac{t}{t_0} + x_0\int_{t_0}^{t} \ln\frac{s}{t_0}\,\underbrace{\frac{ds}{s}}_{=du} = x_0 + x_0\ln\frac{t}{t_0} + \int_0^{\ln\frac{t}{t_0}} u\,du =$$

$$= x_0 + x_0\ln\frac{t}{t_0} + x_0\frac{\ln^2\frac{t}{t_0}}{2}$$

$u = \ln\frac{s}{t_0}$

$du = \frac{1}{\frac{s}{t_0}}\cdot\frac{1}{t_0}\,ds = \frac{ds}{s}$

s	t_0	t
u	0	$\ln\frac{t}{t_0}$

$$x_3(t) = x_0 + \int_{t_0}^{t} F(x_2(s), s)\,ds = x_0 + \int_{t_0}^{t} \frac{x_0 + x_0\ln\frac{s}{t_0} + x_0\frac{\ln^2\frac{s}{t_0}}{2}}{s}\,ds =$$

$$= x_0 + x_0\ln\frac{t}{t_0} + \frac{x_0}{2}\ln^2\frac{t}{t_0} + \frac{x_0}{2}\int_{t_0}^{t}\frac{\ln^2\frac{s}{t_0}}{s}\,ds = x_0 + x_0\ln\frac{t}{t_0} + \frac{x_0}{2}\ln^2\frac{t}{t_0} + \frac{x_0}{\underbrace{2\cdot 3}_{3!}}\ln^3\frac{t}{t_0}$$

$$\int_{t_0}^{t} \frac{\ln^k\frac{s}{t_0}}{s}\,ds = \int_0^{\ln\frac{t}{t_0}} u^k\,du = \frac{\ln^{k+1}\frac{t}{t_0}}{k+1}$$

$k \geq 0$

$u = \ln\frac{s}{t_0}$

$du = \frac{ds}{s}$

s	t_0	t
u	0	$\ln\frac{t}{t_0}$

Претпоставка: (индукцијом)

$$x_n(t) = \sum_{k=0}^{n} \frac{x_0}{k!}\cdot\ln^k\frac{t}{t_0}$$

(базу имамо, $n = 0, 1, \ldots$)

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} F(x_n(s), s)\,ds = x_0 + \int_{t_0}^{t} \frac{x_0\sum_{k=0}^{n}\frac{\ln^k\frac{s}{t_0}}{k!}}{s}\,ds = x_0 + x_0\cdot\sum_{k=0}^{n}\frac{1}{k!}\int_{t_0}^{t}\frac{\ln^k\frac{s}{t_0}}{s}\,ds =$$

$$= x_0 + x_0\cdot\sum_{k=0}^{n}\frac{1}{k!}\cdot\frac{\ln^{k+1}\frac{t}{t_0}}{k+1} = x_0 + x_0\cdot\sum_{k=0}^{n}\frac{\ln^{k+1}\frac{t}{t_0}}{(k+1)!} = x_0 + x_0\cdot\sum_{k=1}^{n+1}\frac{\ln^k\frac{t}{t_0}}{k!} = \sum_{k=0}^{n+1}\frac{x_0}{k!}\ln^k\frac{t}{t_0} \quad ✓$$

$$x_n(t) \to \sum_{k=0}^{\infty} \frac{x_0}{k!}\cdot \ln^k \frac{t}{t_0} = x_0 \cdot \exp\left(\ln \frac{t}{t_0}\right) = \frac{x_0}{t_0}\cdot t$$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

ПР: $x(t) = \frac{x_0}{t_0}\cdot t$

Експонент матрице

$X' = A\cdot X$ линеаран систем дј. са константним коеф. (ЛСДЈКК)

$X : \mathbb{R} \to \mathbb{R}^n$

$t \mapsto X(t)$

$X(t), X'(t) \in \mathbb{R}^n$

$X'(t) = A\cdot X(t)$

$A \in M_n(\mathbb{R})$, $A = [a_{ij}]_{i,j=0}^{n}$

$$X = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

$$\left.\begin{aligned} x_1'(t) &= a_{11}x_1(t) + \dots + a_{1n}x_n(t) \\ &\ \ \vdots \\ x_n'(t) &= a_{n1}x_1(t) + \dots + a_{nn}x_n(t) \end{aligned}\right\}$$

1 једн. 1. реда у $\mathbb{R}^n$
$\Leftrightarrow$
n једн. 1. реда у $\mathbb{R}$

$\boxed{\text{Т}}$ ОР: $X(t) = e^{tA}\cdot c$, $c \in \mathbb{R}^n$, $c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

нема смисла: $\underbrace{c}_{2\times 1}\cdot \underbrace{e^{tA}}_{2\times 2} = X(t)$, $c \in \mathbb{R}^2$

$\exp : M_n(\mathbb{R}) \to M_n(\mathbb{R})$, $\exp(A) = e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$

① Решити систем дј. $X' = AX$, одређивањем e^{tA} у облику степеног реда, ако је:

а) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ б) $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ в) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ г) $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, $a, b \in \mathbb{R}$

$X(t) = e^{tA}\cdot c$, $c \in \mathbb{R}^2$

$e^{tA} = ?$, $A^k = ?$

а) домаћи

б) $e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$

$A^2 = A\cdot A = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 1 & 7 \\ 0 & 8 \end{bmatrix}$$

индукцијом: $A^k = \begin{bmatrix} 1 & 2^k-1 \\ 0 & 2^k \end{bmatrix}$

$$A^4 = (A^2)^2 = \begin{bmatrix} 1 & 15 \\ 0 & 16 \end{bmatrix}$$

База ✓

Корак: $A^{k+1} = A^k \cdot A = \begin{bmatrix} 1 & 1+2^{k+1}-2 \\ 0 & 2^{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 2^{k+1}-1 \\ 0 & 2^{k+1} \end{bmatrix}$ ✓

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{bmatrix} 1 & 2^k-1 \\ 0 & 2^k \end{bmatrix} = \begin{bmatrix} \sum\limits_{k=0}^{\infty} \frac{t^k}{k!} & \sum\limits_{k=0}^{\infty} \frac{t^k(2^k-1)}{k!} \\ 0 & \sum\limits_{k=0}^{\infty} \frac{t^k}{k!} \cdot 2^k \end{bmatrix} = \begin{bmatrix} e^t & e^{2t}-e^t \\ 0 & e^{2t} \end{bmatrix}$$

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} = e^t, \qquad \sum_{k=0}^{\infty} \frac{t^k \cdot 2^k}{k!} = \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} = e^{2t}$$

$$\sum_{k=0}^{\infty} \frac{(2t)^k - t^k}{k!} = \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} - \sum_{k=0}^{\infty} \frac{t^k}{k!} = e^{2t} - e^t$$

ОР: $X(t) = e^{tA} \cdot c = \begin{bmatrix} e^t & e^{2t}-e^t \\ 0 & e^{2t} \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 e^t + c_2(e^{2t}-e^t) \\ c_2 e^{2t} \end{bmatrix} = c_1 \begin{bmatrix} e^t \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} e^{2t}-e^t \\ e^{2t} \end{bmatrix}$

$c \in \mathbb{R}^2$