7 ДЈ са тоталним диференцијалом

$$M(t,x)\,dt + N(t,x)\,dx = 0$$

$$x' = \frac{f(t,x)}{g(t,x)} \iff \frac{dx}{dt} = \frac{f(t,x)}{g(t,x)} \Big/ \cdot \text{„}dt\text{"} \iff$$

$$g(t,x)\,dx = f(t,x)\,dt \iff f(t,x)\,dt - g(t,x)\,dx = 0$$

ДЈ у диференцијалном облику

$\exists F(t,x)$, $dF(t,x) = M(t,x)dt + N(t,x)dx$

претпоставка

$$dF(t,x) = \frac{\partial F}{\partial t}(t,x)\,dt + \frac{\partial F}{\partial x}(t,x)\,dx$$

$$\Rightarrow \left.\begin{aligned} M(t,x) &= \frac{\partial F}{\partial t}(t,x) \Big/ '_x \\ N(t,x) &= \frac{\partial F}{\partial x}(t,x) \Big/ '_t \end{aligned}\right\} \Rightarrow M'_x = \frac{\partial^2 F}{\partial t\,\partial x} = \frac{\partial^2 F}{\partial x\,\partial t} = N'_t$$

$$\text{ТОТ.Д.} \Rightarrow M'_x = N'_t$$

важи и $\boxed{\Leftarrow}$ ако радимо у просто-повезаној области

тамо где је јна деф, где је посматрамо, ...

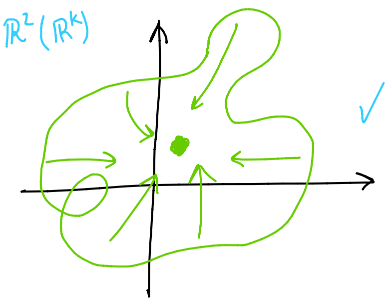
ОР: $0 = M\,dt + N\,dx = dF \Rightarrow F(t,x) = c,\ c \in \mathbb{R}$

имплицитно задато

просто-повезане области (ПП)

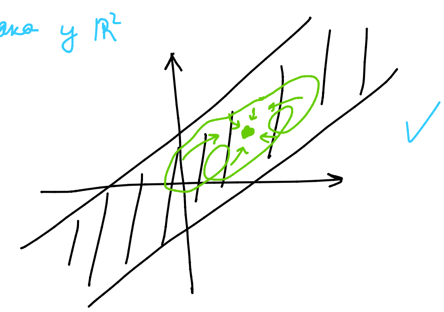
$D \subseteq \mathbb{R}^k$ је ПП $\iff$ свака непрекидна затворена крива у D се може скупити у тачку кроз D и D је повезано

пр. 1) $\mathbb{R}^2$ ($\mathbb{R}^k$)



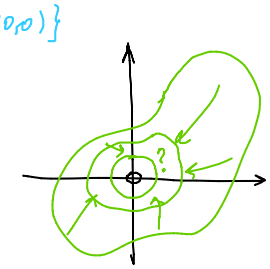
✓

2) трака у $\mathbb{R}^2$



✓

3) $\mathbb{R}^2 \setminus \{(0,0)\}$



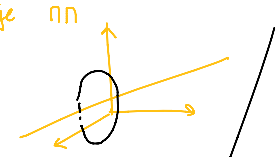
X

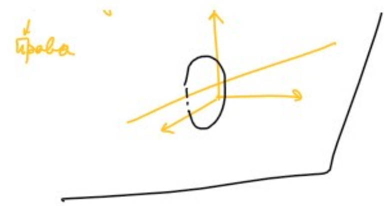
„ПП у $\mathbb{R}^2$ ако нема рупа"

не важи у $\mathbb{R}^k$!

- $\mathbb{R}^3 \setminus \{(0,0,0)\}$ jeсте ПП
- $\mathbb{R}^3 \setminus \ell$ није ПП

права





(1) а) $2t(1+\sqrt{t^2-x})dt-\sqrt{t^2-x}\,dx=0$

б) $(1+x^2\sin t)dt - x\cos^2 t\,dx=0$

в) $(tx^2+3t^2x)dt+(t^3+t^2x)dx=0$

а) $M(t,x)=2t(1+\sqrt{t^2-x})$

$N(t,x)=-\sqrt{t^2-x}$

$$M'_x=2t\cdot\frac{1}{2\sqrt{t^2-x}}(-1)=\frac{-t}{\sqrt{t^2-x}}$$

$$N'_t=-\frac{1}{2\sqrt{t^2-x}}\cdot 2t=\frac{-t}{\sqrt{t^2-x}}$$

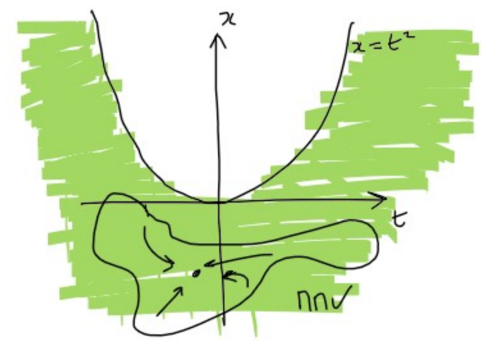
$\Downarrow$

тот. д.

области ?

$t^2-x\geq 0$

$x\leq t^2$



$\Rightarrow \exists F,\quad dF=M\,dt+N\,dx \quad , F=?$

$$\frac{\partial F}{\partial t}=M=2t+2t\sqrt{t^2-x}$$

$$\frac{\partial F}{\partial x}=N=-\sqrt{t^2-x}\quad \Big/\int dx$$

$$F=-\int\sqrt{t^2-x}\,dx=\frac{2}{3}(t^2-x)^{3/2}+\underbrace{c(t)}_{(c\in\mathbb{R}\,x)}\quad , c:\mathbb{R}\to\mathbb{R}$$

$$\Rightarrow \frac{\partial F}{\partial t}=\frac{2}{3}\cdot\frac{3}{2}(t^2-x)^{1/2}\cdot 2t+c'(t)=2t\sqrt{t^2-x}+c'(t)$$

$$\frac{\partial F}{\partial t} = 2t+2t\sqrt{t^2-x}$$

$\Rightarrow c'(t)=2t$ ← ако не добијемо да зависи само од t ⇒ ГРЕШКА (у рачуну)

$c(t)=t^2$

ОР: $\frac{2}{3}(t^2-x)^{3/2}+t^2=c\quad , c\in\mathbb{R}$

18 Интеграциони фактор

$M(t,x)\,dt+N(t,x)\,dx=0$

$M'_x\neq N'_t \Rightarrow$ није тот. д.

Идеја: $\underbrace{\mu(t,x)\cdot\big(M(t,x)\,dt+N(t,x)\,dx\big)}_{\text{јесте тот. д.}}=0 \quad \rightsquigarrow \mu=?$

$\mu(t,x)$ → интеграциони фактор

$$\left.\begin{aligned}\tilde{M} &= \mu\cdot M\\ \tilde{N} &= \mu\cdot N\end{aligned}\right\}\quad \tilde{M}'_x = \tilde{N}'_t$$

$\mu(t,x)$ је често у облику $\mu = \mu(w) = \mu(w(t,x))$

$\mu: \mathbb{R}^2\to\mathbb{R}$, $\mu: \mathbb{R}\to\mathbb{R}$, $w: \mathbb{R}^2\to\mathbb{R}$

$$\tilde{M}'_x = \frac{\partial}{\partial x}(\mu M) = \mu'_x\cdot M + \mu\cdot M'_x = \mu'(w)\cdot w'_x\cdot M + \mu M'_x$$

$$\tilde{N}'_t = \frac{\partial}{\partial t}(\mu N) = \mu'_t\cdot N + \mu\cdot N'_t = \mu'(w)\cdot w'_t\, N + \mu\cdot N'_t$$

$$\Rightarrow\quad \mu'(w)\left(w'_x\cdot M - w'_t\cdot N\right) = \mu\left(N'_t - M'_x\right)$$

$$\mu'(w) = \frac{d\mu}{dw}\qquad\qquad \frac{\mu'(w)}{\mu(w)} = \frac{N'_t - M'_x}{w'_x\cdot M - w'_t\cdot N}$$

$$\frac{d\mu}{\mu} = \boxed{\frac{N'_t - M'_x}{w'_x\cdot M - w'_t\cdot N}}\,dw$$

→ мора да зависи само од w

Циљ: нaћи w!

Често: • $w(t,x) = at + bx$ $(w = x,\ w = t, \dots)$

• $w(t,x) = a\ln|t| + b\ln|x|$

• $w(t,x) = f(t) + g(x)$

• $w(t,x) = f(t)\cdot g(x)$

Проверити $\frac{1}{\mu} = 0$ за изгубљено решење

(2) а) $(q(t) - p(t)x)\,dt - dx = 0$, p, q познате ← $(w = t)$, упоредити са ДЈ линеарне

б) $2tx\ln x\,dt + (t^2 + x^2\sqrt{x^2+1})\,dx = 0$

в) $x(2 - 3tx^2)\,dt - t(1 + tx^2)\,dx = 0$, на области $G = \{x > 0, t > 0\}$, нaћи решење кроз $(2,1)$

г) $(\sqrt{t^2 - x} + 2t)\,dt - dx = 0$, μ у облику $\mu(\underline{t^2 - x})$ $(w = t^2 - x)$

д) $(t+2)\sin x\,dt + 2t\cos x\,dx = 0$ $(w = x)$

$$\left(x' = \frac{dx}{dt} = -\frac{t+2}{t}\cdot \operatorname{tg} x \longrightarrow \text{РП}\right)$$

б) $M'_x = (2tx\ln x)'_x = 2t(1 + \ln x)$

$N'_t = (t^2 + x^2\sqrt{1+x^2})'_t = 2t$ $\quad\#$

б) $M'_x = (2tx\ln x)'_x = 2t(1+\ln x)$

$N'_t = (t^2 + x^2\sqrt{1+x^2})'_t = 2t$ $\quad \nparallel$

$$\frac{d\mu}{\mu} = \frac{2t-(2t+2t\ln x)}{w'_x \cdot M - w'_t \cdot N}\,dw = \frac{-2t\ln x}{w'_x \cdot (2tx\ln x) - \underbrace{w'_t \cdot (t^2+x^2\sqrt{x^2+1})}_{=0}}\,dw = \frac{-2t\ln x}{\frac{1}{x}\cdot 2tx\ln x}\,dw = -dw \;/\int$$

$w'_t = 0 \Rightarrow w = w(x)$

$w = x$ најједноставније

$w = \ln x$

$w = x^2$

нпр. $w = x^2$

$$\frac{d\mu}{\mu} = \frac{-2t\ln x}{2x \cdot 2tx\ln x}\,dw = -\frac{1}{2x^2}\,dw = -\frac{1}{2w}\,dw = -\frac{dw}{2w} \;/\int$$

$\Rightarrow \ln|\mu| = -w + \tilde{c} \;/e^{\square}$

$\mu = c\cdot e^{-w}$ (c=

$\mu(t,x) = \mu(w(t,w)) = c\cdot e^{-w(t,x)} = c\cdot e^{-\ln x} = \frac{c}{x}$ $\quad (c=1)$

$\mu = \frac{1}{x}$

$\frac{1}{\mu} = 0 \Leftrightarrow x = 0$ ✗

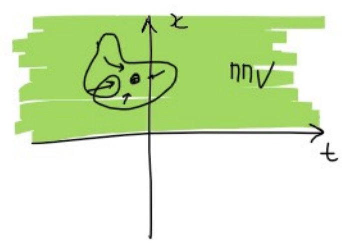
$(\ln x,\ x>0)$

$/\cdot \frac{1}{x} \Rightarrow \underbrace{2t\ln x\,dt}_{M} + \underbrace{\left(\frac{t^2}{x} + x\sqrt{x^2+1}\right)}_{N}dx = 0$ $\quad$ тот. д.?

$M'_x = N'_t$ ✓

област: $x>0$

$x \neq 0$



$\Rightarrow \exists F = ?$

$$\left.\begin{aligned} \frac{\partial F}{\partial t} &= 2t\ln x \\ \frac{\partial F}{\partial x} &= \frac{t^2}{x} + x\sqrt{x^2+1} \end{aligned}\right\} \therefore \quad \text{ОР: } F(t,x) = t^2\ln x + \frac{1}{3}(x^2+1)^{3/2} = c,\ c\in\mathbb{R}$$

в) $x(2-3tx^2)\,dt - t(1+tx^2)\,dx = 0$, на области $G = \{x>0, t>0\}$, наћи решење кроз $(2,1)$

$M'_x = 2 - 9tx^2$

$N'_t = -1 - 2tx^2$

$$\frac{d\mu}{\mu} = \frac{(-1-2tx^2)-(2-9tx^2)}{w'_x \cdot x(2-3tx^2) + w'_t \cdot t(1+tx^2)}\,dw = \frac{-3+7tx^2}{w'_x \cdot x(2-3tx^2) + w'_t \cdot t(1+tx^2)}\,dw$$

$$\frac{d\mu}{\mu} = \frac{(-1-2tx^2)-(2-9tx^2)}{w'_x \cdot x(2-3tx^2)+w'_t \cdot t(1+tx^2)}\,dw = \frac{-3+7tx^2}{w'_x \cdot x(2-3tx^2)+w'_t \cdot t(1+tx^2)}\,dw$$

идеја: w'_x скрати $x \to w'_x = \frac{1}{x}$?

w'_t скрати $t \to w'_t = \frac{1}{t}$?

$$w = a\ln|x| + b\ln|t| = a\ln x + b\ln t \Rightarrow w'_x = \frac{a}{x}$$

$(x,t>0)$

$$w'_t = \frac{b}{t}$$

$$\frac{d\mu}{\mu} = \frac{-3+7tx^2}{a(2-3tx^2)+b(1+tx^2)}\,dw = \frac{-3+7tx^2}{(2a+b)+tx^2(-3a+b)}\,dw = \frac{-3+7tx^2}{-3+7tx^2}\,dw = 1\,dw \quad \Big/ \int$$

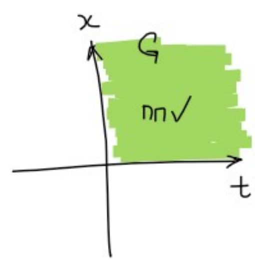
$$2a+b=-3$$
$$-3a+b=7$$

$$\left.\begin{aligned} 5a=-10 \Rightarrow a&=-2 \\ b&=1 \end{aligned}\right\} w = -2\ln x + \ln t$$

$$\Rightarrow \mu = e^{w} = e^{-2\ln x + \ln t} = \frac{t}{x^2}$$

$$\Big/ \cdot \frac{t}{x^2} \Rightarrow \underbrace{\left(2\frac{t}{x} - 3t^2x\right)}_{M} dt - \underbrace{\left(\frac{t^2}{x^2}+t^3\right)}_{N} dx = 0$$

области:



$$M'_x = N'_t$$

$\Rightarrow$ тот. д

$$\left.\begin{aligned} \frac{\partial F}{\partial t} &= 2\frac{t}{x} - 3t^2x \\ \frac{\partial F}{\partial x} &= -\left(\frac{t^2}{x^2}+t^3\right) \end{aligned}\right\} \therefore \overset{\text{ОР:}}{F(t,x) = \frac{t^2}{x} - t^3x = c \in \mathbb{R}}$$

$\Big/ \frac{1}{\mu} = 0 \Leftrightarrow \frac{x^2}{t} = 0 \Leftrightarrow x = 0$?

$(0 \cdot dt - \ldots d0 = 0 ✓)$

$\{x=0\} \cap G = \emptyset$

$\Rightarrow$ није решење

кроз $(2,1)$: $\left.\begin{aligned} t&=2 \\ x&=1 \end{aligned}\right\}$ $\frac{4}{1} - 8\cdot 1 = c \Rightarrow c = -4$

ПР: $\frac{t^2}{x} - t^3x = -4.$