2 Хомогена

$$x' = f\left(\frac{x}{t}\right) \quad , f \in C(a,b)$$

смена: $\frac{x(t)}{t} = y(t)$

$$x = yt \;/' \Rightarrow x' = (yt)' = y' \cdot t + y \cdot 1 = y't + y \quad \Big\} \rightsquigarrow \text{РП}$$

① а) $x' = e^{\frac{x}{t}} + \frac{x}{t}$

б) $x' = -\frac{x^2+t^2}{2xt}$

в) $x' = \frac{x^2-2xt-t^2}{x^2+2xt-t^2}$

г) $t\sin\frac{x}{t} \cdot x' = x \cdot \sin\frac{x}{t} + t$

б) $x' = -\frac{x^2+t^2}{2xt} \stackrel{(:t^2)}{=} -\frac{\left(\frac{x}{t}\right)^2+1}{2\left(\frac{x}{t}\right)}$

$t \neq 0, x \neq 0$

$$\frac{x}{t} = y \Rightarrow x = yt \Rightarrow x' = y't + y$$

$$y't + y = -\frac{y^2+1}{2y}$$

$$y't = -\frac{y^2+1}{2y} - y = -\frac{y^2+1+2y^2}{2y} = -\frac{3y^2+1}{2y}$$

$3y^2+1=0$? ✗

↳ гледамо да ли се изгубило реш? није

$$y' \cdot \frac{2y}{3y^2+1} \stackrel{(\text{РП})}{=} -\frac{1}{t} \;\Big/ \int dt$$

$$\frac{1}{3} \cdot \int \frac{3 \cdot 2y\,dy}{3y^2+1} = -\int \frac{dt}{t} = -\ln|t| + C$$

$$e^{\llcorner} \Big\backslash \; \frac{1}{3}\ln|3y^2+1| = -\ln|t| + C \quad , C \in \mathbb{R}$$

$$|3y^2+1|^{1/3} = |t|^{-1} \cdot e^C$$

$3y^2+1>0$

$$\sqrt[3]{3y^2+1} = \frac{e^C}{|t|} = \frac{C_1}{t} \quad , C_1 \in \mathbb{R}\setminus\{0\}$$

$$(3y^2+1)t^3 = C_2 \quad , C_2 \in \mathbb{R}\setminus\{0\}$$

$(3y^2+1)t^3 = c_2$, $c_2 \in \mathbb{R}\setminus\{0\}$

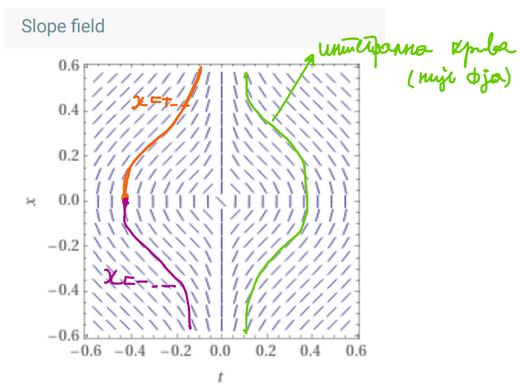
$\Big\downarrow\ y = \frac{x}{t}$

$\left(3\frac{x^2}{t^2}+1\right)t^3 = c_2$

$\underbrace{(3x^2+t^2)t = c_2}$, $c_2 \in \mathbb{R}\setminus\{0\}$

↳ имплицитно задато решење

$x(t) = \dots$ → експлицитно



[3] $x' = f\left(\dfrac{a_1t+b_1x+c_1}{a_2t+b_2x+c_2}\right)$, $f \in C(a,b)$

⇝ РП ∨ ХОМ ∨ [1]

1° $c_1=c_2=0$:

$$x' = f\left(\frac{a_1t+b_1x}{a_2t+b_2x}\right) = f\left(\frac{a_1+b_1\frac{x}{t}}{a_2+b_2\frac{x}{t}}\right) = g\left(\frac{x}{t}\right) \to \text{ХОМ}$$

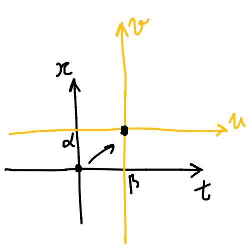
2° $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \neq 0$

$x(t) \rightsquigarrow v(u)$

$\alpha, \beta \in \mathbb{R}$

$x = v+\alpha$

$t = u+\beta$



$$a_1t+b_1x+c_1 = a_1(u+\beta)+b_1(v+\alpha)+c_1 = a_1u+b_1v+\boxed{(a_1\beta+b_1\alpha+c_1)} = 0$$

$$a_2t+b_2x+c_2 = a_2(u+\beta)+b_2(v+\alpha)+c_2 = a_2u+b_2v+\boxed{(a_2\beta+b_2\alpha+c_2)} = 0$$

α, β бирамо као решење система

$$\begin{aligned} a_1\beta+b_1\alpha+c_1 &= 0 \\ a_2\beta+b_2\alpha+c_2 &= 0 \end{aligned} \rightsquigarrow \exists_1\ \alpha,\beta$$

(Крамер)

$x(t) \to v(u)$

$\frac{dx}{dt} \rightsquigarrow \frac{dv}{du} = ?$

x'

$$\frac{dv}{du} = \boxed{\frac{dx}{du}} \cdot \frac{dv}{dx} = \underset{=1}{\frac{dv}{dx}} \cdot \frac{dx}{dt} \cdot \underset{=1}{\frac{dt}{du}} = \frac{dx}{dt} \Rightarrow v' = x'$$

$$\frac{dv}{du} = \boxed{\frac{dx}{du}} \cdot \frac{dv}{dx} = \underset{=1}{\frac{dv}{dx}} \cdot \frac{dx}{dt} \cdot \underset{=1}{\frac{dt}{du}} = \frac{dx}{dt} \Rightarrow v' = x'$$

$$\frac{dx}{du} \to \frac{dx}{dt} \cdot \frac{dt}{du}$$

$$\frac{dv}{dx} = \frac{d(x-\alpha)}{dx} = 1$$

$$\frac{dt}{du} = \frac{d(u+\beta)}{du} = 1$$

$$\Bigg\} \quad v' = f\left(\frac{a_1u+b_1v}{a_2u+b_2v}\right) \to 1^\circ$$

3° $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$

$$a_1b_2 = b_1a_2$$

3.1° ако је неки од њих нула

нпр. $a_1 = 0 \Rightarrow b_1 = 0 \lor a_2 = 0$

$a_1 = b_1 = 0$: $\quad f\left(\dfrac{c_1}{\underbrace{a_2t+b_2x+c_2}_{y(t)}}\right) \rightsquigarrow \boxed{1}$

$a_1 = a_2 = 0$: $\quad f\left(\dfrac{b_1x+c_1}{b_2x+c_2}\right) \rightsquigarrow$ РП

3.2° $\dfrac{b_2}{b_1} = \dfrac{a_2}{a_1} = k$

$$\frac{a_1t+b_1x+c_1}{a_2t+b_2x+c_2} = \frac{a_1t+b_1x+c_1}{ka_1t+kb_1x+c_2} = \frac{\boxed{a_1t+b_1x}+c_1}{k\underbrace{\boxed{(a_1t+b_1x)}}_{y(t)}+c_2} \rightsquigarrow \boxed{1}$$

(2) а) $(x+2t-2)x' = x-t-1$

б) $x' = \dfrac{t+x+4}{t+x-6}$

а) $x' = \dfrac{x-t-1}{x+2t-2 \leftarrow 0?}$

$x+2t-2=0$

$x = -2t+2$

$0 \cdot (-2) = -3t+1 \quad \times$

$$\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = 2-(-1) = 3 \neq 0$$

$x = v+\alpha$

$t = u+\beta$

$$\frac{dv}{du} = \underset{=1}{\frac{dv}{dx}} \cdot \frac{dx}{dt} \cdot \underset{=1}{\frac{dt}{du}} = \frac{dx}{dt}$$

$$\begin{rcases} \alpha-\beta-1 = 0 \\ \alpha+2\beta-2 = 0 \end{rcases} /\cdot 2, \ +$$

$3\alpha-4=0$

$\alpha = \underline{4}$. $\beta = \underline{1}$

$$\frac{dv}{du} = \frac{dv}{dx} \cdot \frac{dx}{dt} \cdot \frac{dt}{du} = \frac{dx}{dt}$$

$3\alpha - 4 = 0$

$\alpha = \frac{4}{3}, \beta = \frac{1}{3}$

$$v' = \frac{\left(v+\frac{4}{3}\right) - \left(u+\frac{1}{3}\right) - 1}{\left(v+\frac{4}{3}\right) + 2\left(u+\frac{1}{3}\right) - 2} = \frac{v-u}{v+2u} \quad \text{(ХОМ)}$$

$$\frac{v}{u} = w(u)$$

$$v = u \cdot w /' \Rightarrow v' = w + u \cdot w'$$

$$w + u \cdot w' = \frac{w-1}{w+2} \Rightarrow u \cdot w' = \frac{w-1}{w+2} - w = \frac{w-1-w^2-2w}{w+2} = -\frac{w^2+w+1}{w+2}$$

$$\frac{w'(w+2)}{w^2+w+1} = -\frac{1}{u} \Big/ \int du \ldots$$

$w^2+w+1 = 0?$

$\times$

$\left(w+\frac{1}{2}\right)^2 + \frac{3}{4} > 0$

$$w(u) \rightsquigarrow v(u) \rightsquigarrow x(t)$$

б) $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$

$$x' = \frac{\boxed{x+t}+4}{\underbrace{\boxed{x+t}}_{y(t)}-6} \rightsquigarrow \boxed{1}$$

$x+t-6=0?$ не може

$\boxed{4}$ Линеарна ДУ 1. реда

$p, g : (a,b) \to \mathbb{R}$ неп.

$$x' + p(t)\,x = g(t)$$

предавања $\longrightarrow$ $x(t) = e^{-\int p\,dt} \cdot \left(C + \int e^{\int p\,dt} \cdot g\,dt \right), \; C \in \mathbb{R}$

$\int p\,dt = \int_{t_0}^{t} p(u)\,du$

исто на оба места

$g(t) \equiv 0 \to$ хомогена лин. дј. 1. реда

$g(t) \not\equiv 0 \to$ нехомогена —||—

③ а) $tx' - x = t^3$

б) $x' + x = \frac{1}{1+e^{2t}}$

в) $x' - 2xt = 6te^{t^2}$

г) $tx' + ax + t^n = 0$, $a \in \mathbb{R}, n \in \mathbb{N}$

в) $p(t) = -2t$

в) $p(t) = -2t$

$q(t) = 6te^{t^2}$ $\quad\quad$ $\int p\,dt = \int -2t\,dt = -t^2 \,(+c)$

$$\int e^{\int p\,dt} \cdot q\,dt = \int \cancel{e^{-t^2}} \cdot 6t \cdot \cancel{e^{t^2}}\,dt = 3t^2 \;(+c)$$

$$x(t) = e^{t^2}\left(c + 3t^2\right), \; c \in \mathbb{R}$$

$$= \underbrace{c \cdot e^{t^2}}_{\text{ОР хомогене},\; q=0} + \underbrace{3t^2 \cdot e^{t^2}}_{\text{ПР нехомогене}}$$

г) $tx' - x = t^3 \;/ : t$

$x' - \frac{1}{t}x = t^2$

$p(t) = -\frac{1}{t}$

$q(t) = t^2$ ...

④ Наћи реш. дј. $x'\sin t - x\cos t = -\frac{\sin^2 t}{t^2}$ за које важи $\lim\limits_{t\to\infty} x(t) = 0$.

I начин: по формули

II начин: $(x \cdot \sin t)' = x' \cdot \sin t + x \cdot \cos t$?

$(fg)' = f'g + fg'$

$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

$$\left(\frac{x}{\sin t}\right)' = \frac{x'\sin t - x\cos t}{\sin^2 t}$$

$$\frac{x'\sin t - x\cos t}{\sin^2 t} = -\frac{1}{t^2}$$

$$\left(\frac{x}{\sin t}\right)' = -\frac{1}{t^2} \;/ \int dt$$

$$\frac{x}{\sin t} = \frac{1}{t} + c$$

$$x = \frac{\sin t}{t} + c\sin t, \; c \in \mathbb{R}$$

$\lim\limits_{t\to\infty} x(t) = 0$

↳ није Кошијев услов

$$\frac{\sin t}{t} \xrightarrow[t\to\infty]{} 0$$

$$\Rightarrow \; c\sin t \xrightarrow[t\to\infty]{} 0 \;? \;\Rightarrow c = 0$$

ПР: $x(t) = \frac{\sin t}{t}$.

5 Бернулијева дј.

$x' + p(t)x = q(t)\cdot x^{\alpha}$, $\alpha \in \mathbb{R}$ $\qquad p,q : (a,b) \to \mathbb{R}$

$\alpha = 0$: линеарна

$\alpha = 1$, линеарна

$\alpha \notin \{0,1\}$ надаље

смена: $y(t) = x(t)^{1-\alpha}$

$$y' = (x^{1-\alpha})' = (1-\alpha)\cdot x^{-\alpha}\cdot x' \Rightarrow x^{-\alpha}\cdot x' = \frac{y'}{1-\alpha}$$

$\big\downarrow \cdot x^{-\alpha}$

$$\underbrace{x'x^{-\alpha}} + p(t)\cdot x^{1-\alpha} = q(t)$$

$$\frac{y'}{1-\alpha} + p(t)y = q(t)$$

$$y' + (1-\alpha)p(t)y = (1-\alpha)q(t) \longrightarrow \text{лин.}$$

5) а) $x' = \frac{x}{t} - x^2$

б) $x' + \frac{x}{t} = x^2\frac{\ln t}{t}$

в) $tx' - 2t\sqrt{x} = 4x$

а) $x' - \frac{x}{t} = -x^2$, $t \neq 0$

$$\begin{bmatrix} p(t) = -\frac{1}{t} \\ q(t) = -1 \end{bmatrix} \qquad y = x^{1-\alpha} = x^{1-2} = x^{-1} = \frac{1}{x}$$

$$x^{\alpha} = x^2, \ \alpha = 2 \qquad y' = (-1)x^{-2}\cdot x' = -\frac{x'}{x^2}$$

$$\underbrace{x'\cdot x^{-2}}_{-y'} - \frac{1}{t}\cdot\underbrace{\frac{1}{x}}_{y} = -1$$

$$-y' - \frac{y}{t} = -1 \Rightarrow y' + \frac{y}{t} = 1.$$

$$p(t) = \frac{1}{t}$$

$$q(t) = 1$$

$$\int p\,dt = \int\frac{dt}{t} = \ln|t|$$

$$\int e^{\int p\,dt}\cdot q\,dt = \int e^{\ln|t|}\cdot 1\,dt = \int |t|\,dt = \int \operatorname{sgn} t\cdot t\,dt = \operatorname{sgn} t\cdot\int t\,dt = \operatorname{sgn} t\cdot\frac{t^2}{2} = \frac{t|t|}{2}.$$

$$y(t) = e^{-\int p\,dt}\left(c + \int e^{\int p\,dt} \cdot q\,dt\right) = e^{-\ln|t|} \cdot \left(c + \frac{t|t|}{2}\right) = \frac{1}{|t|}\left(c + \frac{t|t|}{2}\right) = \frac{c}{|t|} + \frac{t}{2} = \frac{c_1}{t} + \frac{t}{2}$$

$c, c_1 \in \mathbb{R}$

$$x(t) = \frac{1}{y(t)} = \frac{1}{\frac{c_1}{t} + \frac{t}{2}} = \frac{2t}{2c_1 + t^2} = \frac{2t}{c_2 + t^2}, \quad c_2 \in \mathbb{R}$$

B) $t x' - 2t\sqrt{x} = 4x \quad / : t$

$$x' - 2\sqrt{x} = 4\frac{x}{t}$$

$$x' - 4\frac{x}{t} = 2\underbrace{\boxed{\sqrt{x}}}_{x^\alpha}$$

$\sqrt{x} = x^\alpha, \quad \alpha = \frac{1}{2} \ldots$