12 - четвртак 23.12.2021.

(1) $(y+u)u'_x + y\,u'_y = x-y \quad , \; u\big|_{y=1} = x+1$

(КЛ)

$u\big|_{y=1} = x+1 \iff u(x,1) = x+1 \iff (s,1,s+1) \in \Gamma(u)$

$$x' = y+u \qquad x_0(s) = s$$
$$y' = y \qquad y_0(s) = 1$$
$$u' = x-y \qquad u_0(s) = s+1$$

$$X = \begin{bmatrix} x \\ y \\ u \end{bmatrix}$$

$$X' = AX, \quad A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

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>> A=[0 1 1; 0 1 0; 1 -1 0]

A =

     0     1     1
     0     1     0
     1    -1     0

>> [P D] = jordan(A)

P =

    -1     1     1
     0     1     0
     1     0     1


D =

    -1     0     0
     0     1     0
     0     0     1
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$$A = PDP^{-1}$$

$$e^{tA} = Pe^{tD}P^{-1}$$

$$\text{ОР}: \; X(t) = P\cdot e^{tD}\cdot c = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{-t} & & \\ & e^{t} & \\ & & e^{t} \end{bmatrix} \cdot c = \begin{bmatrix} -e^{-t} & e^{t} & e^{t} \\ 0 & e^{t} & 0 \\ e^{-t} & 0 & e^{t} \end{bmatrix} \cdot c$$

$$X(t,s) = \begin{bmatrix} \cdots \end{bmatrix} \cdot c(s)$$

$$X(0,s) = \begin{bmatrix} x_0(s) \\ y_0(s) \\ u_0(s) \end{bmatrix} = \begin{bmatrix} s \\ 1 \\ s+1 \end{bmatrix}$$

$$X(0,s) = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot c(s)$$

$$c(s) = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} s \\ 1 \\ s+1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} s \\ 1 \\ s+1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ 2 \\ 2s \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ s \end{bmatrix}$$

$$X(t,s) = \begin{bmatrix} -e^{-t} & e^{t} & e^{t} \\ 0 & e^{t} & 0 \\ e^{-t} & 0 & e^{t} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ s \end{bmatrix} = \begin{bmatrix} e^{t}(1+s) - e^{-t} \;\text{"}x\text{"} \\ e^{t} = y \\ se^{t} + e^{-t} \;\text{"}u\text{"} \end{bmatrix}$$

решење: $(x,y,u) = (e^{t}(1+s) - e^{-t},\; e^{t},\; se^{t} + e^{-t})$ – површ у $\mathbb{R}^3$

стиц: можемо и експлицитно да је изразимо

$$e^t = y \Rightarrow t = \ln y$$

$$e^{-t} = e^{-\ln y} = \frac{1}{y}$$

$$x = y(1+s) - \frac{1}{y} \Rightarrow s = -1 + \frac{x}{y} + \frac{1}{y^2}$$

$$u(x,y) = se^t + e^{-t} = \left(-1 + \frac{x}{y} + \frac{1}{y^2}\right)\cdot y + \frac{1}{y} = \underline{\underline{x - y + \frac{2}{y}}}$$

провера: $(y+u)\,u'_x + y\,u'_y = x - y$

$u'_x = 1$

$u'_y = -1 - \frac{2}{y^2}$

$$\left(y + x - y + \frac{2}{y}\right)\cdot 1 + y\cdot\left(-1 - \frac{2}{y^2}\right) = x - y$$

$$x + \frac{2}{y} - y - \frac{2}{y} = x - y \quad ✓$$

$$u\big|_{y=1} = \left(x - y + \frac{2}{y}\right)\Big|_{y=1} = x - 1 + \frac{2}{1} = x + 1$$

(2) $y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0$, $(s, s, s^4) \subseteq \Gamma(u)$

(хл), али посматрамо као (кл)

$$\left.\begin{array}{l} a_1(x,y,u) = y \\ a_2(x,y,u) = -x \\ c(x,y,u) = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} x' = y \\ y' = -x \\ u' = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x = c_1\cos t + c_2\sin t \\ y = c_2\cos t - c_1\sin t \\ u = c_3 \end{array}$$

$x_0(s) = y_0(s) = s$

$u_0(s) = s^4$

$$\Rightarrow c_1 = c_2 = s$$

$$c_3 = s^4$$

површ: $\left(s(\cos t + \sin t),\ s(\cos t - \sin t),\ s^4\right)$

стиц: $x^2 + y^2 = s^2(\cos^2 t + \sin^2 t + 2\cos t\sin t + \cos^2 t + \sin^2 t - 2\cos t\sin t) = 2s^2$

$$(x^2+y^2)^2 = 4s^4 = 4u \Rightarrow u = \frac{(x^2+y^2)^2}{4}$$

приметимо:

$x = c_1\cos t + c_2\sin t$

$y = c_2\cos t - c_1\sin t$

$$x^2+y^2 = c_1^2(\cos^2 t+\sin^2 t) + c_2^2(\cos^2 t+\sin^2 t) = c_1^2+c_2^2 = const$$

$\Rightarrow x^2+y^2$ је први интеграл система

тврдим да је $u = \varphi(x^2+y^2)$, $\varphi \in C^1(\mathbb{R})$ решење ПДЈ:

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0 \quad , \quad \frac{\partial u}{\partial x} = \varphi'(x^2+y^2)\cdot 2x$$

$$\frac{\partial u}{\partial y} = \varphi'(x^2+y^2)\cdot 2y$$

$$y\cdot\varphi'(x^2+y^2)\cdot 2x - x\cdot\varphi'(x^2+y^2)\cdot 2y = 0 \ \checkmark$$

Метода првих интеграла

идеја: у систему (*) наћи $\boxed{\text{скупове}}$ где је решење ПДЈ константно

↳ први интеграли система (*)

$\psi_1, \dots, \psi_{n-1}$

(КЛ) $u(x_1,\dots,x_n)$ се тражи $\rightsquigarrow$ сис. кар. је реда $n+1$ $\to$ потребно је n првих интеграла $\psi_1,\dots,\psi_n$

ОР: $\boxed{\varphi(\psi_1,\dots,\psi_n)=0}$, $\varphi\in C^1(\mathbb{R}^n)$

↳ имплицитно

(ХЛ) $u(x_1,\dots,x_n)$ се тражи $\rightsquigarrow$ сис. кар. је реда n $\to$ потребно је $n-1$ првих интеграла $\psi_1,\dots,\psi_{n-1}$

ОР: $\boxed{u=\varphi(\psi_1,\dots,\psi_{n-1})}$, $\varphi\in C^1(\mathbb{R}^{n-1})$

↳ експлицитно

(3) $$(x^2-y^2-z^2)\frac{\partial z}{\partial x} + 2xy\frac{\partial z}{\partial y} = 2xz$$

(КЛ)

$$x' = x^2-y^2-z^2$$
$$y' = 2xy$$
$$z' = 2xz$$

$z(x,y)$

↳ 2 прва интеграла

$$\int\frac{y'}{y}\,dt = \int\frac{dy}{y} = \ln|y|+c$$

$$\frac{y'}{z'} = \frac{2xy}{2xz} = \frac{y}{z} \Rightarrow \frac{y'}{y} = \frac{z'}{z} \Big/ \int dt \Rightarrow \ln|y| = \ln|z| + c_1$$

$$\ln|y| - \ln|z| = c_1$$

$\frac{dy}{dz}$ (као смена: посматрамо $y(z)$)

$$\ln\left|\frac{y}{z}\right| = c_1 \ \ldots \ \frac{y}{z} = c_2 \Rightarrow \psi_1(x,y,z) = \frac{y}{z}$$

$$y = c_2 z \Rightarrow \left.\begin{aligned} x' &= x^2 - c_2^2 z^2 - z^2 \\ z' &= 2xz \end{aligned}\right\} \quad \frac{dx}{dz} = \frac{x'}{z'} = \frac{x^2 - c_2^2 z^2 - z^2}{2xz} = \frac{x}{2z} - \frac{z(c_2^2+1)}{2x}$$

$$\hookrightarrow \frac{1}{x} = x^{-1} = x^{\alpha}$$

Бернулијева за $\alpha = -1$, смена: $u(z) = x(z)^2$

⋮

$$x(z)^2 = u(z) = cz - z^2(1 + c_2^2) = cz - z^2\left(1 + \frac{y^2}{z^2}\right) = cz - z^2 - y^2$$

$$c_2 \to \frac{y}{z}$$

$$c = \frac{x^2 + y^2 + z^2}{z} = \psi_2(x,y,z)$$

ОР: $\Phi(\psi_1, \psi_2) = 0$, $\Phi \in C^1(\mathbb{R}^2)$

$$\Phi\left(\frac{y}{z}, \frac{x^2+y^2+z^2}{z}\right) = 0$$

4. $(4y - 3z)\frac{\partial u}{\partial x} + (4x - 2z)\frac{\partial u}{\partial y} + (2y - 3x)\frac{\partial u}{\partial z} = 0$ са условом $u(x,y,1) = (x+2)^2 - (y+3)^2$

$u(x,y,z)$ $(X\cap)$

↳ 2 прва интеграла

$$\begin{aligned} x' &= 4y - 3z \\ y' &= 4x - 2z \\ z' &= 2y - 3x \end{aligned}$$

[домаћи: методом карактеристика $X' = AX$, e^{tA} ...]

идеја: линеарне комбинације

$\alpha, \beta \in \mathbb{R}$

$$\alpha(4y - 3z) + \beta(4x - 2z) = 2y - 3x$$

$$x(4\beta) + y(4\alpha) + \underbrace{z(-3\alpha - 2\beta)}_{=0} = 2y - 3x$$

$$\begin{aligned} 4\alpha &= 2 \\ 4\beta &= -3 \\ -3\alpha - 2\beta &= 0 \end{aligned}$$

$$\alpha = \frac{1}{2}, \quad \beta = -\frac{3}{4}$$

$$\frac{1}{2}\cdot(4y - 3z) - \frac{3}{4}\cdot(4x - 2z) = 2y - 3x$$

$$\int x'\,dt = x + c$$

$$\frac{1}{2}\cdot(4y-3z)-\frac{3}{4}\cdot(4x-2z)=2y-3x$$

$$\frac{1}{2}x'-\frac{3}{4}y'=z' \;\Big/\int dt$$

$$\int x'\,dt = x+c$$

$$\frac{1}{2}x-\frac{3}{4}y=z+c \;/\cdot 4$$

$$2x-3y-4z=c_1=\psi_1(x,y,z)$$

друкчије лин. комб:

$$\alpha\cdot x\cdot(4y-3z)+\beta y(4x-2z)=z\cdot(2y-3x)$$

$$xy\underbrace{(4\alpha+4\beta)}_{=0}+xz(-3\alpha)+yz(-2\beta)=-3xz+2yz$$

$$\alpha=1$$
$$\beta=-1$$

$$x\cdot(4y-3z)-y(4x-2z)=z\cdot(2y-3x)$$

$$x\cdot x'-yy'=zz' \;\Big/\int dt$$

$$\int xx'\,dt=\frac{x^2}{2}+c$$

$$\frac{x^2}{2}-\frac{y^2}{2}=\frac{z^2}{2}+c \;/\cdot 2$$

$$\left(\frac{x^2}{2}\right)'=\frac{2\cdot x\cdot x'}{2}=xx'$$

$$x^2-y^2-z^2=c_2=\psi_1(x,y,z)$$

$$\cancel{\int xy'\,dt=xy+c}$$

$$(xy)'=x'y+xy'$$

$$\int(x'y+y'x)\,dt=xy+c$$

ОР: $u=\varphi(\psi_1,\psi_2)$, $\varphi\in C^1(\mathbb{R}^2)$

дат Кошијев: $u(x,y,1)=(x+2)^2-(y+3)^2 \leftarrow z=1$

$$u(x,y,z)=\varphi(2x-3y-4z,\;x^2-y^2-z^2)\leftarrow z=1$$

$$u(x,y,1)=\varphi(2x-3y-4,\;x^2-y^2-1)$$
$$\parallel$$
$$(x+2)^2-(y+3)^2$$
$$\parallel$$
$$x^2-y^2+4x-6y-5$$

наш посао: наћи φ

$$\varphi(p_1,p_2)=2p_1+p_2+4$$

Кошијево реш: $u=2\psi_1+\psi_2+4=4x-6y-8z+x^2-y^2-z^2+4$

$$=(x+2)^2-(y+3)^2-(z+4)^2+25$$

(5) $x(y^2-z^2)\frac{\partial u}{\partial x}-y(x^2+z^2)\frac{\partial u}{\partial y}+z(x^2+y^2)\frac{\partial u}{\partial z}=0$, $u|_{\ldots}=(u+z)^2$

(5) $x(y^2-z^2)\frac{\partial u}{\partial x} - y(x^2+z^2)\frac{\partial u}{\partial y} + z(x^2+y^2)\frac{\partial u}{\partial z} = 0 \quad , \ u|_{x=1} = (y+z)^2$

$$u(1,y,z) = (y+z)^2$$
$$(1,y,z,(y+z)^2) \subseteq \Gamma(u) \subseteq \mathbb{R}^4$$

OP? (X,n) $u(x,y,z) \rightsquigarrow$ 2 прва инт.

$$x' = x(y^2-z^2)$$
$$y' = -y(x^2+z^2)$$
$$z' = z(x^2+y^2)$$

- $\alpha x(y^2-z^2) - \beta y(x^2+z^2) = z(x^2+y^2)$ ✗
- $\alpha x \cdot x(y^2-z^2) - \beta y \cdot y(x^2+z^2) = z \cdot z(x^2+y^2)$

$$\alpha = \beta = -1 \checkmark$$

$$-xx' - yy' = zz' \ / \int dt$$

$$-\frac{x^2}{2} - \frac{y^2}{2} = \frac{z^2}{2} + c \ / \cdot (-2)$$

$$\psi_1 = x^2+y^2+z^2$$

- $\frac{\alpha}{x} x(y^2-z^2) - \frac{\beta}{y} \cdot y(x^2+z^2) = \frac{1}{z} \cdot z(x^2+y^2)$

$$\alpha(y^2-z^2) - \beta(x^2+z^2) = x^2+y^2$$

$$\alpha = 1$$
$$\beta = -1$$

$$(\ln|x|)' = \frac{1}{|x|} \cdot \operatorname{sgn} x \cdot x' = \frac{x'}{x}$$

$$\frac{1}{x}x' - \frac{1}{y}y' = \frac{1}{z}z' \ / \int dt$$

$$\ln|x| - \ln|y| = \ln|z| + c$$
$$\vdots$$
$$\frac{yz}{x} = \psi_2(x,y,z)$$

OP: $u = \varphi(\psi_1, \psi_2)$, $\varphi \in C^1(\mathbb{R}^2)$

Košijevo? $\rightsquigarrow \varphi = ?$

$$u\big|_{x=1} = \varphi\left(\psi_1\big|_{x=1},\ \psi_2\big|_{x=1}\right)$$

$$(y+z)^2 = \varphi(1+y^2+z^2,\ yz) = (1+y^2+z^2) + 2\cdot(yz) - 1$$

$$\varphi(p,q) = p + 2q - 1$$

$$u(x,y,z) = \psi_1 + 2\psi_2 - 1 = x^2 + y^2 + z^2 + \frac{2yz}{x} - 1$$

[домаћи: проверити да u задов. јну]