Кошијев проблем: $x^{(n)} + a_{n-1}(t)x^{(n-1)} + \dots + a_1(t)x' + a_0(t)x = f(t)$

услови:
$$\left.\begin{aligned} x(t_0) &= x_0 \\ x'(t_0) &= x_1 \\ x''(t_0) &= x_2 \\ &\vdots \\ x^{(n-1)}(t_0) &= x_{n-1} \end{aligned}\right\} \text{ n услова (јна)}$$

↳ у истој тачки

није КП:
- $x(0)=1, x(1)=2$
- $x'(0)=0, x(1)=2$

① Решити Кошијев проблем $x''' + x'' = 0$ (3. реда)

$x(0)=1$

$x'(0)=0$

$2 \to x''(0)=1$

$\lambda^3 + \lambda^2 = 0$

$\lambda^2(\lambda+1)=0$

$\hookrightarrow \underbrace{0,0}_{1,t}, \underbrace{-1}_{e^{-t}}$

ОР: $x(t) = c_1 + c_2 t + c_3 e^{-t}, \ c_i \in \mathbb{R}$

$x(0) = c_1 + c_3 = 1$

$x'(t) = c_2 - c_3 e^{-t}$

$x'(0) = c_2 - c_3 = 0$

$x''(t) = c_3 e^{-t}$

$x''(0) = c_3 = 1$

$$\left.\begin{aligned} c_2 = c_3 = 1 \\ c_1 = 0 \end{aligned}\right\} \quad x_K(t) = t + e^{-t}$$

нехомогени случај: $f \neq 0$

$x^{(n)} + a_{n-1}x^{(n-1)} + \dots + a_1 x' + a_0 x = f(t)$

у посебним случајевима: $f(t) = e^{\alpha t}\cdot(P_n(t)\cdot\cos\beta t + Q_m(t)\cdot\sin\beta t)$, $\alpha,\beta \in \mathbb{R}$

P_n - пол. ст. n

Q_m - пол. ст. m

s - вишеструкост броја $\alpha \pm i\beta$ као решења каракт. јне

$x_p(t) = t^s \cdot e^{\alpha t}\cdot(R_k(t)\cos\beta t + T_k(t)\sin\beta t)$

R_k, T_k - пол. ст. k

$k = \max\{n, m\}$

② a) $x''' - x'' + x' - x = t^2 + t$

ОР: $x(t) = x_H(t) + x_p(t)$

$x_H(t) = c_1 e^t + c_2\cos t + c_3 \sin t, \ c_1, c_2, c_3 \in \mathbb{R}$

$\lambda^3 - \lambda^2 + \lambda - 1 = 0$

$(\lambda-1)(\lambda^2+1) = 0$

$\hookrightarrow 1, \pm i$

$x_p(t) = ?$

$\alpha = 0 \quad (e^{\alpha t} = 1)$

$\beta = 0 \quad (\cos\beta t = 1, \sin\beta t = 0)$

$$\left.\begin{array}{l} P_n(t) = t + t^2 \rightsquigarrow n = 2 \\ Q_m(t) - \text{[illegible]} \quad (m = 0) \end{array}\right\} K = \max\{2, 0\} = 2.$$

$s = ? \qquad \alpha \pm i\beta = 0 \pm i0 = 0 \notin \{1, \pm i\} \Rightarrow s = 0$

$$x_p(t) = \underset{1}{t^0} \cdot \underset{1}{e^{0t}} \cdot \big(R_2(t) \cdot \underset{1}{\cos 0t} + T_2(t) \cdot \underset{0}{\sin 0t}\big) = R_2(t) = at^2 + bt + c$$

$$x_p''' - x_p'' + x_p' - x_p = t + t^2$$

$$\left.\begin{array}{l} x_p' = 2at + b \\ x_p'' = 2a \\ x_p''' = 0 \end{array}\right\}$$

$$0 - 2a + 2at + b - at^2 - bt - c = t + t^2$$

$$\left.\begin{array}{l} -a = 1 \\ 2a - b = 1 \\ -2a + b - c = 0 \end{array}\right\} \quad \begin{array}{l} a = -1 \\ b = -3 \\ c = -1 \end{array}$$

$$x_p(t) = -t^2 - 3t - 1$$

б) $x''' - x'' + x' - x = \cos t + e^{2t}$ → није у одг. облику

$1, \pm i$

$f_1(t) = \cos t \rightsquigarrow x_{p_1}(t)$

$f_2(t) = e^{2t} \rightsquigarrow x_{p_2}(t)$

ОР: $x(t) = x_H(t) + x_{p_1}(t) + x_{p_2}(t)$

$$x_H(t) = c_1 e^t + c_2 \cos t + c_3 \sin t, \; c_i \in \mathbb{R}$$

лин.

$L(x) = f_1(t) + f_2(t)$

$L(x_{p_1}) = f_1(t)$

$L(x_{p_2}) = f_2(t)$

$L(x_{p_1} + x_{p_2}) = L(x_{p_1}) + L(x_{p_2}) = f_1(t) + f_2(t)$

лин.

f_1: $\alpha = 0$

$\beta = 1$

$$\left.\begin{array}{l} P_n(t) \equiv 1, \; n = 0 \\ Q_m(t) \equiv 0, \; m = -\infty \end{array}\right\} K = 0 \Rightarrow R_0 = c_1, \; T_0 = c_2$$

$\alpha \pm i\beta = 0 \pm i = \pm i \Rightarrow \underline{\underline{s = 1}} \Rightarrow x_{p_1}(t) = t \cdot (c_1 \cos t + c_2 \sin t)$

$c_1 = c_2 = -\frac{1}{4}$

$$\alpha \pm i\beta = 0 \pm i = \pm i \Rightarrow \underline{\underline{s=1}} \Rightarrow x_{p_1}(t) = t\cdot(c_1\cos t + c_2\sin t)$$

$$\vdots \quad c_1 = c_2 = -\frac{1}{4}$$

f_2: $\alpha = 2$

$\beta = 0$

$$\left.\begin{array}{l} P_n(t) \equiv 1 \Rightarrow n=0 \\ Q_m(t) - \text{било шта } (m=0) \end{array}\right\} k=0 \Rightarrow R_0(t)\equiv c_1,\ T_0(t)\equiv c_2$$

$$\alpha \pm i\beta = 2 \pm i0 = 2 \Rightarrow s=0 \Rightarrow x_{p_2}(t) = e^{2t}\cdot(c_1\cos 0t + c_2\sin 0t) = c_1 e^{2t} \dots \quad c_1 = \frac{1}{5}$$

в) $x'' - x = \sin^2 t$

$$f(t) = \sin^2 t = \frac{1-\cos 2t}{2} = \underbrace{\frac{1}{2}}_{x_{p_1}} - \underbrace{\frac{\cos 2t}{2}}_{x_{p_2}}$$

г) $x'' - 4x' + 5x = (\sin t + 2\cos t)\cdot e^{2t}$

$\boxed{2\pm i}$

$\alpha = 2$

$\beta = 1$

$$\left.\begin{array}{l} P_n(t)\equiv 2 \\ Q_m(t) \equiv 1 \end{array}\right\} n=m=0 \Rightarrow k=0 \Rightarrow \begin{array}{l} R_0(t)\equiv c_1 \\ T_0(t) = c_2 \end{array} \Bigg\} x_p(t) = t\cdot e^{2t}\cdot(c_1\cdot\cos t + c_2\sin t)$$

$\vdots$

$$\alpha \pm i\beta = \boxed{2\pm i} \Rightarrow \underline{\underline{s=1}}$$

д) $x'' - 2x' + x = \dfrac{e^t}{t}$ – није у одг. облику → другачије

$(\lambda-1)^2 = 0$

$1,1 \rightsquigarrow e^t, t\cdot e^t$

идеја: $x_p(t) = e^t\cdot g(t)$

$$x_p'(t) = e^t\cdot g(t) + e^t\cdot g'(t) = e^t(g(t)+g'(t))$$

$$x_p''(t) = e^t\cdot(g(t)+g'(t)+g'(t)+g''(t)) = e^t\cdot(g(t)+2g'(t)+g''(t))$$

$$e^t\cdot(g''+2g'+g) - 2\cdot e^t(g+g') + e^t g = \frac{e^t}{t} \quad /:e^t$$

$$g''+2g'+g-2g-2g'+g = \frac{1}{t}$$

$$g''=\frac{1}{t} \Rightarrow g' = \ln|t| + c_1 \Rightarrow g(t) = t\cdot\ln|t| - t + c_1 t + c_2$$

$c_1 = 1$, $c_2 = 0$ $\Rightarrow$ $g(t) = t\cdot\ln|t|$

$$x_p(t) = e^t\cdot t\cdot\ln|t|$$

OP: $x(t) = c_1e^t + c_2te^t + e^t t\ln|t|$, $c_i\in\mathbb{R}$

Парцијалне ДЈ 1. реда

ОДЈ $x(t)$ → еволуција у времену, нпр. $\ddot{x} = F$ – II Н. закон

ПДЈ $u(x_1,\dots,x_n) = ?$ → еволуција у простору и времену, нпр. $u_t = u_{xx}$

↓ једн. провођења топлоте

$u(t,x)$ – интензитет топлоте

t – време

нотација: $u_t = u'_t = \frac{\partial u}{\partial t}$

$u_{xx} = u''_{xx} = \frac{\partial^2 u}{\partial x^2}$

$v''_{xy} = v_{xy} = \frac{\partial^2 v}{\partial x\partial y}$

$u_t = u_{xx}$ → 2. реда

1. реда ⟶ $F\left(x_1,\dots,x_n,u,\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n}\right) = 0$ → 1. реда

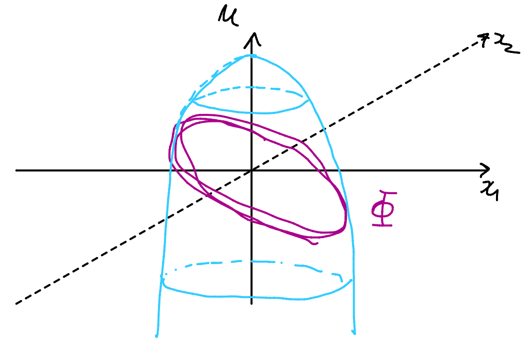
→ Квазилинеарна (КЛ): $\sum_{j=0}^{n} a_j(x_1,\dots,x_n,u)\frac{\partial u}{\partial x_j} = c(x_1,\dots,x_n,u)$

[линеарна: $\sum_{j=0}^{n} a_j(x_1,\dots,x_n)\frac{\partial u}{\partial x_j} = c(x_1,\dots,x_n)$]

→ хомогена линеарна (ХЛ): $\sum_{j=0}^{n} a_j(x_1,\dots,x_n)\frac{\partial u}{\partial x_j} = 0$ $(c=0)$

Кошијев проблем: наћи решење које садржи задату фигуру Φ.

$$\Phi \subseteq \underbrace{\Gamma(u)}_{\text{график}}$$



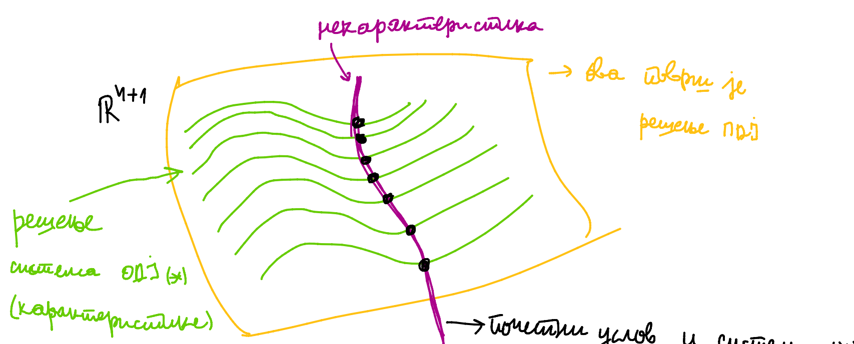
Метода карактеристика

(КЛ) ⇒ систем карактеристика (*)

$$\sum_{j=0}^{n} a_j(x_1,\dots,x_n,u)\frac{\partial u}{\partial x_j} = c(x_1,\dots,x_n,u) \Rightarrow \left.\begin{aligned} x_j'(t) &= a_j(x_1,\dots,x_n,u),\ \forall j \\ u'(t) &= c(x_1,\dots,x_n,u) \end{aligned}\right\} (*)$$

(ХЛ) ⇒ систем карактеристика (*)

$$\sum_{j=0}^{n} a_j(x_1,\dots,x_n)\frac{\partial u}{\partial x_j} = 0 \Rightarrow x_j'(t) = a_j(x_1,\dots,x_n),\ \forall j \quad (*)$$



③ $\underbrace{u'_x + u'_y + 2u = 1 + u^2}_{(КЛ)}$, $u(\sin x, x+x^2) = x$

↳ Кошијев услов

$$\underbrace{1}_{a_1}\cdot u'_x + \underbrace{1}_{a_2}\cdot u'_y = \underbrace{1-2u+u^2}_{c} \Rightarrow \left.\begin{aligned} x'(t) &= 1 \\ y'(t) &= 1 \\ u'(t) &= 1-2u+u^2 \end{aligned}\right\}$$

$$x' = 1 \Rightarrow x(t) = t + c_1$$

$$y' = 1 \Rightarrow y(t) = t + c_2$$

$$u' = (1-u)^2 \Rightarrow \frac{u'}{(1-u)^2} = 1 \quad \overset{(РП)}{/\int}$$

$$\int \frac{du}{(1-u)^2} = t + c_3$$

$$\frac{1}{1-u} = t + c_3$$

$$1 - u = \frac{1}{t+c_3} \Rightarrow u = 1 - \frac{1}{t+c_3} = \frac{t+c_3-1}{t+c_3}$$

$$u(\underbrace{\sin x}_{x}, \underbrace{x+x^2}_{y}) = \underbrace{x}_{u} \to \text{крива у } \mathbb{R}^3$$

$$\gamma(s) = (\underline{\underline{\sin s}}, \underline{\underline{s+s^2}}, \underline{\underline{s}}) \subseteq \Gamma(u) \subseteq \mathbb{R}^3$$

→ некарактеристика

$$c_1, c_2, c_3 \rightsquigarrow c_1(s), c_2(s), c_3(s)$$

$$x(t,s) = t + c_1(s)$$

$$y(t,s) = t + c_2(s)$$

$$u(t,s) = \frac{t + c_3(s) - 1}{t + c_3(s)}$$

почетни услов: $x(0,s) = x_0(s) = \sin s$

$y(0,s) = y_0(s) = s + s^2$

$u(0,s) = u_0(s) = s$

$$\sin s = x(0,s) = c_1(s)$$

$$s + s^2 = y(0,s) = c_2(s)$$

$$s = u(0,s) = \frac{c_3(s)-1}{c_3(s)} \Rightarrow s\,c_3(s) = c_3(s) - 1 \Rightarrow c_3(s)(1-s) = 1 \Rightarrow c_3(s) = \frac{1}{1-s}$$

површ (параметарски задата — s и t):

$$(x(t,s), y(t,s), u(t,u)) = \left(t + \sin s,\ t + s + s^2,\ \frac{t + \frac{s}{1-s}}{t + \frac{1}{1-s}} \right) \checkmark$$