(1) $y' = Ay$

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix}, \quad \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda = 2$$

$$k = 4$$

$$(A-\lambda E)v = 0 \Rightarrow \dots \quad v \in \mathrm{Lin}\left\{ \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{= v_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{= v_4} \right\}^{\nearrow \dim = 2} \Rightarrow m = 2 \Rightarrow 2 \text{ Жорд. блока}$$

$$\underbrace{\begin{bmatrix} 2 & 1 & & \\ & 2 & & \\ & & 2 & 1 \\ & & & 2 \end{bmatrix}}_{2+2} \vee \underbrace{\begin{bmatrix} 2 & 1 & & \\ & 2 & 1 & \\ & & 2 & \\ & & & 2 \end{bmatrix}}_{3+1}$$

$\varphi(\lambda) = (\lambda - 2)^4$

$\mu(\lambda) = (\lambda-2)^k, \ k \leq 4$

$\mu(A) = 0 \Rightarrow (A-2E)^k = 0$

$A - 2E \neq 0$

$(A-2E)^2 \neq 0$

$(A-2E)^3 = 0 \Rightarrow \deg \mu = 3 \Rightarrow$ ред највећег блока је 3 $\Rightarrow J = \begin{bmatrix} \boxed{\begin{matrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{matrix}} & \\ & \boxed{2} \end{bmatrix}$

v_1, v_4 – сопс. век. $\Rightarrow$ треба још 2 уопштена

уопштени за v_4:

$(A-2E)v = v_4$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 0 = 0 \\ 0 = 1 \ \times \\ d = 0 \\ a = 0 \end{matrix}$$

$\Rightarrow$ систем нема решења $\Rightarrow v_4$ нема уопштене сопс. век.

$\Rightarrow v_4$ одговара блоку 1x1

уопштени за v_1:

$$(A-2E)v_2 = v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{matrix} 0 = 0 \\ 0 = 0 \\ d = 1 \\ a = 0 \end{matrix} \Rightarrow v_2 = \begin{bmatrix} 0 \\ b \\ c \\ 1 \end{bmatrix} \xrightarrow{b=c=0} v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$v_1 \quad v_4$

$v_1 \to v_2 \to v_3 \to v_3$ је уопштени за v_2

$$(A-2E)v_3 = v_2 = \begin{bmatrix} 0 \\ 0 \\ \dots \end{bmatrix}$$

$a = 0$

$$(A-2E)\gamma_3 = \gamma_2 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$$\begin{matrix}0=0\\0=0\\d=0\\a=1\end{matrix} \Rightarrow \gamma_3 = \begin{bmatrix}1\\b\\c\\0\end{bmatrix} \xrightarrow{b=c=0} \gamma_3 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

$$J = \begin{bmatrix}\boxed{\begin{matrix}2&1&\\&2&1\\&&2\end{matrix}}&\\&\boxed{2}\end{bmatrix}$$

$$T = \begin{bmatrix}\gamma_1\downarrow & \gamma_2\downarrow & \gamma_3\downarrow & \gamma_4\downarrow\end{bmatrix} = \begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix} \qquad (\det T \neq 0)$$

$$e^{xJ} = \begin{bmatrix}e^{xB}&\\&e^{2x}\end{bmatrix} = e^{2x}\cdot\begin{bmatrix}1&x&\frac{x^2}{2}&0\\0&1&x&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$$J = \begin{bmatrix}\boxed{B}_{3\times3}&\\&2\end{bmatrix}$$

$$B = \begin{bmatrix}2&1&\\&2&1\\&&2\end{bmatrix} = \underbrace{\begin{bmatrix}2&&\\&2&\\&&2\end{bmatrix}}_{D} + \underbrace{\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}}_{N}$$

$$DN = ND \Rightarrow e^{xB} = e^{xD}\cdot e^{xN} = \begin{bmatrix}e^{2x}&&\\&e^{2x}&\\&&e^{2x}\end{bmatrix}\cdot\left(E + xN + \frac{x^2}{2}N^2\right) = e^{2x}\cdot\begin{bmatrix}1&x&\frac{x^2}{2}\\0&1&x\\0&0&1\end{bmatrix}$$

$$N^2 = \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}, \qquad N^3 = 0$$

OP: $y(x) = T\cdot e^{xJ}\cdot c,\ c\in\mathbb{R}^4$

② $y' = Ay$

$$A = \begin{bmatrix}2&1&1\\0&1&-1\\0&1&3\end{bmatrix}, \qquad \varphi(\lambda) = -(\lambda-2)^3$$

$\lambda_1=\lambda_2=\lambda_3=2 \Rightarrow k=3$

$$(A-2E)\gamma = 0 \rightsquigarrow \gamma = a\cdot\begin{bmatrix}1\\0\\0\end{bmatrix} + b\cdot\begin{bmatrix}0\\1\\-1\end{bmatrix} \Rightarrow \dim\operatorname{Ker}(A-2E) = 2 \Rightarrow m = 2$$

$\Rightarrow$ 2 блока

$\begin{bmatrix} 0 \end{bmatrix} \quad \begin{bmatrix} -1 \end{bmatrix}$ $\Rightarrow$ 2 блока

Једина могућност: $J = \begin{bmatrix} \boxed{\begin{matrix} 2 & 1 \\ & 2 \end{matrix}} & \\ & \boxed{2} \end{bmatrix}$

$2+1$

Провера: $A-2E \neq 0$, $(A-2E)^2 = 0 \Rightarrow \mu(\lambda) = (\lambda-2)^2 \Rightarrow \deg \mu = 2$

$$e^{xJ} = \begin{bmatrix} e^{x\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}} & \\ & e^{2x} \end{bmatrix} = e^{2x} \cdot \begin{bmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$T = ?$ имамо 2 сопс. век $\gamma_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ и $\gamma_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

треба нам 1 уопштени (γ_3) - који од γ_1 или γ_2 одговора?

$(A-2E)\gamma_3 = \gamma_1$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$b + c = 1$
$-b - c = 0$
$b + c = 0$ ↯

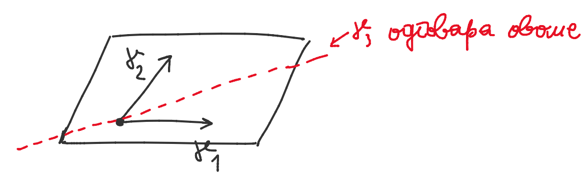
$(A-2E)\gamma_3 = \gamma_2$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

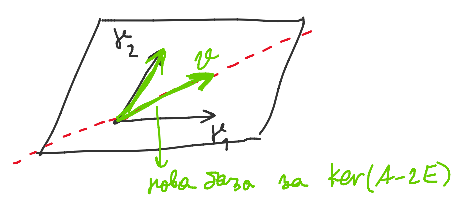
$b + c = 0$
$-b - c = 1$
$b + c = -1$ ↯

шта сад?

сопствени потпростор:



мењамо базу:



$v = ?$, $v \in Lin\{\gamma_1, \gamma_2\}$ и да v има уопштени сопс. век γ_3

$(A-2E)\gamma_3 = v$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = v = \alpha \cdot \gamma_1 + \beta \cdot \gamma_2 = \begin{bmatrix} \alpha \\ \beta \\ -\beta \end{bmatrix}$$

$$\left.\begin{array}{l} b+c=\alpha \\ -b-c=\beta \quad \cdot(-1) \\ \cancel{b+c=-\beta} \end{array}\right\} \quad \alpha=-\beta=1$$

$$v = \gamma_1 - \gamma_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \rightsquigarrow \mathcal{L}in\{\gamma_1, \gamma_2\} = \mathcal{L}in\{v, \gamma_2\} = \operatorname{Ker}(A-2E)$$

$$\left.\begin{array}{l} b+c=1 \\ -b-c=-1 \end{array}\right\} c = 1-b \rightsquigarrow \gamma_3 = \begin{bmatrix} a \\ b \\ 1-b \end{bmatrix} \xrightarrow{a=b=0} \gamma_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$J = \begin{bmatrix} \boxed{\begin{matrix} 2 & 1 \\ & 2 \end{matrix}} & \\ & \boxed{2} \end{bmatrix}$$

$$T = \begin{bmatrix} \boxed{v\downarrow \; \gamma_3\downarrow} & \boxed{\gamma_2\downarrow} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \quad (\det T \neq 0) \quad \Rightarrow \text{OP}: \; y(x) = Te^{xJ} \cdot c, \; c \in \mathbb{R}^3$$

3) $y' = Ay$

$$A = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \qquad \begin{array}{l} \lambda_1 = \lambda_2 = 2 \\ \lambda_{3/4} = 1 \pm i \end{array}$$

$\underline{\lambda_1 = \lambda_2 = 2:}$ $(k=2)$

$$(A-2E)\gamma_1 = 0 \Rightarrow \gamma_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$m=1 \Rightarrow$ један уопштени

и један блок

$$\rightsquigarrow \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \text{ у } J$$

$$(A-2E)\gamma_2 = \gamma_1$$

$$\vdots$$

$$\gamma_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$\underline{\lambda_{3/4} = 1 \pm i:}$ $\quad \alpha + i\beta = 1 + i, \; \alpha = \beta = 1$

$$\rightsquigarrow \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \text{ у } J$$

$$(A - (1+i)E)\gamma_3 = 0 \rightsquigarrow \gamma_3 = \begin{bmatrix} 1 \\ -2 \\ -1+i \\ -2i \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 0 \end{bmatrix} + i \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{1 \quad 1} & \boxed{} \end{bmatrix} = \begin{bmatrix} \boxed{1 \quad 0} & \boxed{1 \quad 1} \end{bmatrix}$$

$$\Rightarrow \; J = \begin{bmatrix} 1 & 1 & & \\ -1 & 1 & & \\ & & 2 & 1 \\ & & & 2 \end{bmatrix} \quad и \quad T = \begin{bmatrix} 1 & 0 & 1 & 1 \\ -2 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & -2 & 0 & 0 \end{bmatrix}$$

$$e^{xJ} = \begin{bmatrix} e^{x\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}} & \\ & e^{x\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}} \end{bmatrix} = \begin{bmatrix} e^{x} \cdot R_x & \\ & e^{2x}\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} e^{x}\cos x & e^{x}\sin x & & \\ -e^{x}\sin x & e^{x}\cos x & & \\ & & e^{2x} & xe^{2x} \\ & & & e^{2x} \end{bmatrix}$$

ОР: $y(x) = Te^{xJ} \cdot c \; , \; c \in \mathbb{R}^4$

(4) $y' = Ay$

$$A = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \qquad \lambda_{1/2} = \lambda_{3/4} = \pm i$$

$$\lambda_1 = i \; , \; k=2 \longrightarrow \alpha + i\beta \Rightarrow \alpha = 0 \; , \; \beta = 1 \rightsquigarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ & & 0 & 1 \\ & & -1 & 0 \end{bmatrix} \quad \vee \quad J = \begin{bmatrix} 0 & 1 & 1 & \\ -1 & 0 & & 1 \\ & & 0 & 1 \\ & & -1 & 0 \end{bmatrix}$$

↳ 2 Жорданова блока

↳ 1 Жорданов блок

$(A - \lambda_1 E)\gamma_1 = 0$

$$\therefore \; \gamma_1 = \begin{bmatrix} 0 \\ -i \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + i \cdot \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \underbrace{\dim_{\mathbb{C}}}_{\text{комплексна дим.}} Ker(A - \lambda_1 E) = 1 \Rightarrow m = 1 \Rightarrow 1 \text{ Жорд. бл.}$$

$$\Rightarrow J = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

ако би се добила 2 сопс. век: γ_1 и γ_2

$$\Rightarrow m = 2 \Rightarrow J = \begin{bmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{bmatrix} \; и \; T = \begin{bmatrix} Re\,\gamma_1 \downarrow & Im\,\gamma_1 \downarrow & Re\,\gamma_2 \downarrow & Im\,\gamma_2 \downarrow \end{bmatrix} \; , \; e^{xJ} = \begin{bmatrix} e^{x\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} & \\ & e^{x\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} \end{bmatrix}$$

$$\Rightarrow m=2 \Rightarrow J = \begin{bmatrix} \begin{smallmatrix}0&1\\-1&0\end{smallmatrix} & \\ & \begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\end{bmatrix} \text{ и } T = \begin{bmatrix} \operatorname{Re}\gamma_1\downarrow & \operatorname{Im}\gamma_1\downarrow & \operatorname{Re}\gamma_2\downarrow & \operatorname{Im}\gamma_2\downarrow \end{bmatrix}, \quad e^{xJ} = \begin{bmatrix} e^{x\left[\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right]} & \\ & e^{x\left[\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right]} \end{bmatrix}$$

γ_2 - уточнени за γ_1

$$(A-\lambda_1 E)\gamma_2 = \gamma_1$$

$$\therefore \gamma_2 = \begin{bmatrix}1\\0\\-1-i\\0\end{bmatrix} = \begin{bmatrix}1\\0\\-1\\0\end{bmatrix} + i\cdot\begin{bmatrix}0\\0\\-1\\0\end{bmatrix}$$

$$\Rightarrow T = \begin{bmatrix} \operatorname{Re}\gamma_1\downarrow & \operatorname{Im}\gamma_1\downarrow & \operatorname{Re}\gamma_2\downarrow & \operatorname{Im}\gamma_2\downarrow \end{bmatrix} = \begin{bmatrix}0&0&1&0\\0&-1&0&0\\0&0&-1&-1\\1&0&0&0\end{bmatrix} \qquad (\det T \neq 0)$$

$$J = \begin{bmatrix} A & E_2 \\ 0 & A \end{bmatrix} \qquad \left(e^{xJ} \neq \begin{bmatrix} e^{xA} & e^{x E_2} \\ & e^{xA} \end{bmatrix} \right)$$

$$A = \begin{bmatrix}0&1\\-1&0\end{bmatrix}, \quad E_2 = \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$J = \underbrace{\begin{bmatrix}A&0\\0&A\end{bmatrix}}_{D} + \underbrace{\begin{bmatrix}0&E_2\\0&0\end{bmatrix}}_{N}, \qquad DN = \begin{bmatrix}A&0\\0&A\end{bmatrix}\begin{bmatrix}0&E_2\\0&0\end{bmatrix} = \begin{bmatrix}0&A\\0&0\end{bmatrix} = \begin{bmatrix}0&E_2\\0&0\end{bmatrix}\begin{bmatrix}A&0\\0&A\end{bmatrix} = ND$$

$$\Downarrow$$

$$e^{xJ} = e^{xD}\cdot e^{xN}$$

$$N^2 = 0 \Rightarrow e^{xN} = E + x\cdot N = \begin{bmatrix}1&0&x&0\\0&1&0&x\\0&0&1&0\\0&0&0&1\end{bmatrix} = \begin{bmatrix}E_2 & xE_2\\0 & E_2\end{bmatrix}$$

$$e^{xD} = \begin{bmatrix} e^{xA} & 0 \\ 0 & e^{xA}\end{bmatrix} = \begin{bmatrix} R_x & 0 \\ 0 & R_x \end{bmatrix} \qquad \Rightarrow e^{xJ} = \begin{bmatrix} R_x & xR_x \\ 0 & R_x \end{bmatrix}$$

$$\text{ОР}: \ Y(x) = Te^{xJ}\cdot c = \begin{bmatrix}0&0&1&0\\0&-1&0&0\\0&0&-1&-1\\1&0&0&0\end{bmatrix}\cdot\begin{bmatrix}\cos x & \sin x & x\cos x & x\sin x\\ -\sin x & \cos x & -x\sin x & x\cos x\\ 0&0&\cos x&\sin x\\ 0&0&-\sin x&\cos x\end{bmatrix}\cdot c, \quad c\in\mathbb{R}^4$$