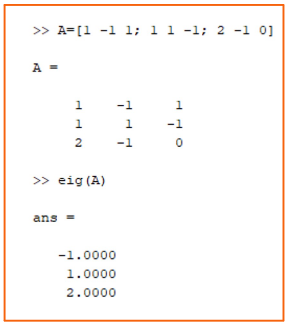
5 - уторак 30.03.2021.

① $y_1' = y_1 - y_2 + y_3$

$y_2' = y_1 + y_2 - y_3$

$y_3' = 2y_1 - y_2$

$$y' = Ay, \quad A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix}, \quad \det(A - \lambda E) = 0$$



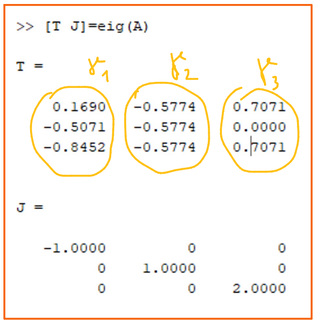
```
>> A=[1 -1 1; 1 1 -1; 2 -1 0]

A =

     1    -1     1
     1     1    -1
     2    -1     0

>> eig(A)

ans =

   -1.0000
    1.0000
    2.0000
```

← сопствене вредности матрице A

$$\lambda_1 = -1, \ \lambda_2 = 1, \ \lambda_3 = 2$$

$$A = T\underset{\sim}{\dot{J}}T^{-1}$$

$$\dot{J} = \begin{bmatrix} -1 & & \\ & 1 & \\ & & 2 \end{bmatrix}, \quad T = \begin{bmatrix} \gamma_1\downarrow & \gamma_2\downarrow & \gamma_3\downarrow \end{bmatrix}$$



```
>> [T J]=eig(A)

T =

    0.1690   -0.5774    0.7071
   -0.5071   -0.5774    0.0000
   -0.8452   -0.5774    0.7071


J =

   -1.0000         0         0
         0    1.0000         0
         0         0    2.0000
```

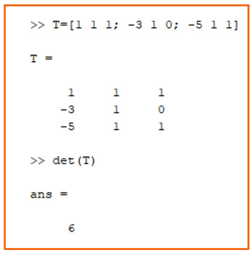
T - матрица промене базе са нормираним векторима

J - дијагонална J~A

$(A - \lambda_1 E)\gamma_1 = 0$ ↴ $\quad \lambda_2$ ↴ $\quad \lambda_3$ ↴

$$\gamma_1 = \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \gamma_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

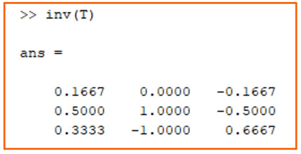
$$T = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 1 & 0 \\ -5 & 1 & 1 \end{bmatrix}$$



```
>> T=[1 1 1; -3 1 0; -5 1 1]

T =

     1     1     1
    -3     1     0
    -5     1     1

>> det(T)

ans =

     6
```

→ детерминанта матрице T

↗ инверз матрице T



```
>> inv(T)

ans =

    0.1667    0.0000   -0.1667
    0.5000    1.0000   -0.5000
    0.3333   -1.0000    0.6667
```

$T^{-1} = ?$

$$T^{-1} = \frac{1}{\det T} \cdot \operatorname{Adj} T = \frac{1}{6} \cdot \begin{bmatrix} +\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} -3 & 0 \\ -5 & 1 \end{vmatrix} & +\begin{vmatrix} -3 & 1 \\ -5 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ -5 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ -5 & 1 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ -3 & 0 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ -3 & 1 \end{vmatrix} \end{bmatrix}^T = \frac{1}{6} \cdot \begin{bmatrix} 1 & 3 & 2 \\ 0 & 6 & -6 \\ -1 & -3 & 4 \end{bmatrix}^T = \frac{1}{6} \begin{bmatrix} 1 & 0 & -1 \\ 3 & 6 & -3 \\ 2 & -6 & 4 \end{bmatrix}$$

$$A = T\dot{J}T^{-1}$$

$$\text{OP}: \ y(x) = e^{xA} \cdot c = Te^{x\dot{J}}\underbrace{T^{-1} \cdot c}_{c_1} = T \cdot e^{x\dot{J}} \cdot c_1. \qquad c, c_1 \in \mathbb{R}^3$$

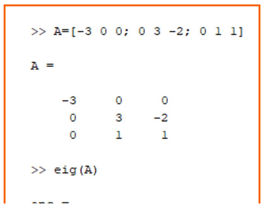
$$e^{x\dot{J}} = \begin{bmatrix} e^{-x} & & \\ & e^{x} & \\ & & e^{2x} \end{bmatrix}$$

② $y_1' = -3y_1$

$y_2' = 3y_2 - 2y_3$

$y_3' = y_2 + y_3$

$\begin{bmatrix} -3 & 0 & 0 \end{bmatrix}$

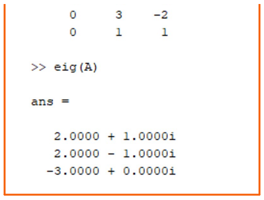


```
>> A=[-3 0 0; 0 3 -2; 0 1 1]

A =

    -3     0     0
     0     3    -2
     0     1     1

>> eig(A)
```

$y_3' = \quad y_2 + y_3$

$$A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

$\lambda_1 = -3 \quad , \quad \lambda_{2/3} = 2 \pm i$



```
     0     3    -2
     0     1     1

>> eig(A)

ans =

   2.0000 + 1.0000i
   2.0000 - 1.0000i
  -3.0000 + 0.0000i
```

$$J = \begin{array}{cc} \lambda_1=-3 & \lambda_2=2+i \\ \begin{bmatrix} -3 & & \\ & \boxed{\begin{matrix} 2 & 1 \\ -1 & 2 \end{matrix}} \end{bmatrix} \end{array}$$

$$\alpha + i\beta \leftrightarrow \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

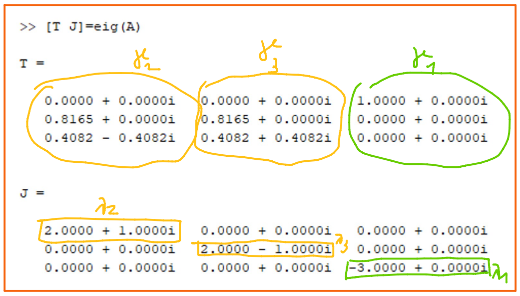
$\lambda_1 = -3 \rightsquigarrow v_1$

$\lambda_2 = 2+i \rightsquigarrow v_2 \rightsquigarrow \mathrm{Re}\, v_2 \ , \mathrm{Im}\, v_2$

$(A - \lambda_1 E) v_1 = 0 \rightsquigarrow v_1$

$(A - \lambda_2 E) v_2 = 0 \rightsquigarrow v_2$

$$T = \begin{bmatrix} v_1 \downarrow & \mathrm{Re}\, v_2 \downarrow & \mathrm{Im}\, v_2 \downarrow \end{bmatrix}$$



```
>> [T J]=eig(A)

T =

   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i
   0.8165 + 0.0000i   0.8165 + 0.0000i   0.0000 + 0.0000i
   0.4082 - 0.4082i   0.4082 + 0.4082i   0.0000 + 0.0000i


J =

   2.0000 + 1.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   2.0000 - 1.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i  -3.0000 + 0.0000i
```

↗ за комплексне нам eig не даје оно што желимо баш

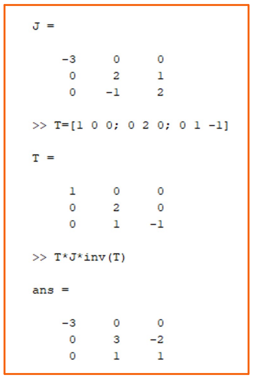
$\lambda_3 = \overline{\lambda_2} \Rightarrow v_3 = \overline{v_2}$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 0 \\ 2 \\ 1-i \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + i \cdot \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

провера да је све ок:



```
J =

    -3     0     0
     0     2     1
     0    -1     2

>> T=[1 0 0; 0 2 0; 0 1 -1]

T =

     1     0     0
     0     2     0
     0     1    -1

>> T*J*inv(T)

ans =

    -3     0     0
     0     3    -2
     0     1     1
```

↗ * је множење матрица

$A = T \cdot J \cdot T^{-1}$

$A = T \underline{J} T^{-1}$

$$e^{xJ} = \begin{bmatrix} e^{-3x} & \\ & \boxed{e^{2x} \cdot R_x} \end{bmatrix}$$

$$\xleftarrow{\text{дијагонала}} \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \Rightarrow e^{x \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}} = \begin{bmatrix} e^{B_1 x} & 0 \\ 0 & e^{B_2 x} \end{bmatrix}$$

$$\begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}^k = \begin{bmatrix} B_1^k & 0 \\ 0 & B_2^k \end{bmatrix}$$

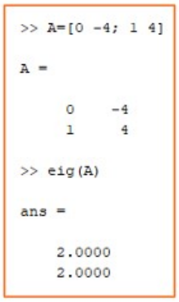
подсетник: $e^{x \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}} = e^{\alpha x} \cdot \begin{bmatrix} \cos(\beta x) & \sin(\beta x) \\ -\sin(\beta x) & \cos(\beta x) \end{bmatrix} = e^{\alpha x} \cdot R_{\beta x}$

ОР: $y(x) = e^{xA} \cdot c = T e^{xJ} T^{-1} \cdot c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} e^{-3x} & 0 & 0 \\ 0 & e^{2x}\cos x & e^{2x}\sin x \end{bmatrix} \cdot c_1 \quad , \quad c, c_1 \in \mathbb{R}^3$

$$\text{OP}: \; y(x) = e^{xA}\cdot c = T e^{xJ}\cdot \underbrace{T^{-1}\cdot c}_{c_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} e^{-1x} & 0 & 0 \\ 0 & e^{2x}\cos x & e^{2x}\sin x \\ 0 & -e^{2x}\sin x & e^{2x}\cos x \end{bmatrix} \cdot c_1 \quad , \; c, c_1 \in \mathbb{R}^3$$

③ $y' = Ay$

$$A = \begin{bmatrix} 0 & -4 \\ 1 & 4 \end{bmatrix}$$



```
>> A=[0 -4; 1 4]

A =

     0    -4
     1     4

>> eig(A)

ans =

    2.0000
    2.0000
```

$\lambda_1 = \lambda_2 = 2$

$k=2$ - алгебарска вишеструкост

За J имамо могућности: $J = \begin{bmatrix} \boxed{2} & 0 \\ 0 & \boxed{2} \end{bmatrix}$ $\vee$ $J = \begin{bmatrix} \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} \end{bmatrix}$

↳ 2 Жорданова бл. (дијагонално)

↳ 1 Жорданов бл.

$(A - \lambda_1 E)\gamma_1 = 0$

$$\begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\dim \operatorname{Ker}(A - \lambda_1 E) = \dim \mathcal{L}in\left\{ \begin{bmatrix} -2b \\ b \end{bmatrix} \middle| b \in \mathbb{R} \right\} = 1$

$\Rightarrow$ $-2a - 4b = 0$, ~~$a + 2b = 0$~~ $\Big\}$ $a = -2b$

$$\gamma_1 = \begin{bmatrix} -2b \\ b \end{bmatrix} \underset{\text{нпр. } b=1}{=} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

↝ имамо само 1 сопс. век.

(увек $m \le k$)

$m = 1$ - геометријска вишеструкост

$\Downarrow$

имамо 1 Жорданов блок

$\Downarrow$

$$J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

$A = TJT^{-1}$

$T = ?$

γ_2 - други базни вектор ће бити уопштени сопс. век.

$(A - \lambda_1 E)\gamma_2 = \gamma_1$

$$\begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

~~$-2a - 4b = -2$~~

$a + 2b = 1$

$$a = 1 - 2b \rightsquigarrow \gamma_2 = \begin{bmatrix} 1-2b \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

нпр. $b=0 \Rightarrow \gamma_2=\begin{bmatrix}1\\0\end{bmatrix}$.

$$T=\begin{bmatrix}\gamma_1\downarrow & \gamma_2\downarrow\end{bmatrix}=\begin{bmatrix}-2&1\\1&0\end{bmatrix}$$

$$e^{xJ}=e^{x(D+N)}=e^{xD}\cdot e^{xN}=\begin{bmatrix}e^{2x}&0\\0&e^{2x}\end{bmatrix}\cdot\left(E+x\cdot\frac{N}{1!}\right)=e^{2x}\cdot\begin{bmatrix}1&x\\0&1\end{bmatrix}=\begin{bmatrix}e^{2x}&xe^{2x}\\0&e^{x}\end{bmatrix}.$$

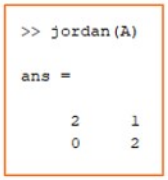
$$J=\begin{bmatrix}2&1\\0&2\end{bmatrix}=\underset{\overset{\parallel}{D}}{\begin{bmatrix}2&0\\0&2\end{bmatrix}}+\underset{\overset{\parallel}{N}}{\begin{bmatrix}0&1\\0&0\end{bmatrix}}$$

$$DN=2EN=2N=N\cdot 2E=ND \Rightarrow e^{xJ}=e^{xD}\cdot e^{xN}$$

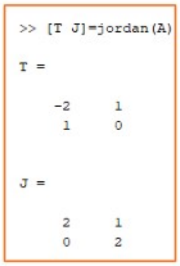
N-нилпотентна $(\exists k, N^k=0)$

$$N^2=\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

ОР: $y(x)=e^{xA}\cdot c=T\cdot e^{xJ}\cdot c_1 \quad , c, c_1\in\mathbb{R}^2$



```
>> jordan(A)

ans =

     2     1
     0     2
```



```
>> [T J]=jordan(A)

T =

    -2     1
     1     0


J =

     2     1
     0     2
```

→ даје Жорданову форму J и матрицу прелаcка T

④ $y'=Ay$

$$A=\begin{bmatrix}2&0&0&0\\0&2&1&0\\0&0&2&0\\1&0&0&2\end{bmatrix} \Rightarrow \lambda_1=\lambda_2=\lambda_3=\lambda_4=2$$

$k=4$

могућности за J:

$$\begin{bmatrix}\boxed{2}&&&\\&\boxed{2}&&\\&&\boxed{2}&\\&&&\boxed{2}\end{bmatrix},\quad \begin{bmatrix}\boxed{\begin{matrix}2&1\\&2\end{matrix}}&&\\&\boxed{2}&\\&&\boxed{2}\end{bmatrix},\quad \begin{bmatrix}\boxed{\begin{matrix}2&1\\&2\end{matrix}}&\\&\boxed{\begin{matrix}2&1\\&2\end{matrix}}\end{bmatrix}$$

$$\begin{bmatrix}\boxed{\begin{matrix}2&1&\\&2&1\\&&2\end{matrix}}&\\&\boxed{2}\end{bmatrix},\quad \begin{bmatrix}\boxed{\begin{matrix}2&1&&\\&2&1&\\&&2&1\\&&&2\end{matrix}}\end{bmatrix}$$

$(A-2E)\gamma=0$

$$\begin{bmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\\1&0&0&0\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

~~$0=0$~~
$c=0$
~~$0=0$~~
$a=0$

$$\rightsquigarrow \gamma=\begin{bmatrix}0\\b\\0\\d\end{bmatrix}=b\cdot\begin{bmatrix}0\\1\\0\\0\end{bmatrix}+d\cdot\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$$\dim\operatorname{Ker}(A-2E)=\dim\left\{\begin{bmatrix}0\\b\\0\\d\end{bmatrix} \,\middle|\, b,d\in\mathbb{R}\right\}=2$$

$\Rightarrow m=2 \Rightarrow 2$ Жорданова блока $\Rightarrow \begin{bmatrix}2&1&&\\&2&1&\\&&2&\\&&&2\end{bmatrix} \vee \begin{bmatrix}2&1&&\\&2&&\\&&2&1\\&&&2\end{bmatrix}$

μ - минимални полином од A

φ - каракт. полином

$$\varphi(\lambda)=\det(A-\lambda E)=(\lambda-2)^4 \quad , \mu \mid \varphi \Rightarrow \mu=(\lambda-2)^k \quad , k\leq 4$$

$$A-2E\neq 0$$

$(A-2E)^2=0 \Rightarrow \mu(\lambda)=(\lambda-2)^2 \Rightarrow \deg\mu = st\,\mu = 2$ ⇒ ред највећег Жордановог блока је 2

$$\Rightarrow \text{мора бити} \quad \ddot{J}=\begin{bmatrix}2&1&&\\&2&&\\&&2&1\\&&&2\end{bmatrix}$$

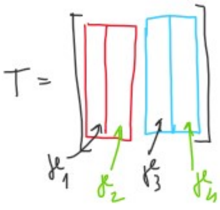
$$e^{xJ}=e^{x\begin{bmatrix}B&0\\0&B\end{bmatrix}}=\begin{bmatrix}e^{xB}&0\\0&e^{xB}\end{bmatrix}=\begin{bmatrix}e^{2x}&xe^{2x}&0&0\\0&e^{2x}&0&0\\0&0&e^{2x}&xe^{2x}\\0&0&&e^{2x}\end{bmatrix}$$

$$B=\begin{bmatrix}2&1\\&2\end{bmatrix}$$

$T=?$ сопств. век. $\gamma_1=\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$, $\gamma_3=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$

сваки има свој Ж. блок

$$J=\begin{bmatrix}2&1&&\\&2&&\\&&2&1\\&&&2\end{bmatrix}$$



γ_2 - уопштени за γ_1

$(A-2E)\gamma_2=\gamma_1$

~~0 = 0~~
$c = 1$
~~0 = 0~~
$a = 0$

$$\Rightarrow \gamma_2=\begin{bmatrix}0\\b\\1\\d\end{bmatrix} \xrightarrow{b=d=0} \gamma_2=\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$

γ_4 - уопштени за γ_3

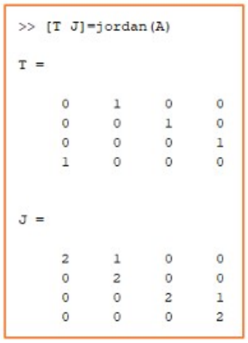
$(A-2E)\gamma_4=\gamma_3$

~~0 = 0~~
$c = 0$
~~0 = 0~~
$a = 1$

$$\Rightarrow \gamma_4=\begin{bmatrix}1\\b\\0\\d\end{bmatrix} \xrightarrow{b=d=0} \gamma_4=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

$$T=\begin{bmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}$$

($\det T\neq 0$) проверити, јер T мора бити инвертибилна



```
>> [T J]=jordan(A)

T =

     0     1     0     0
     0     0     1     0
     0     0     0     1
     1     0     0     0


J =

     2     1     0     0
     0     2     0     0
     0     0     2     1
     0     0     0     2
```

ОР: $y(x)=T\cdot e^{x\dot{J}}\cdot c$, $c\in\mathbb{R}^4$