Нехомогени ЛСДЈКК

$$Y'(x) = \underbrace{AY(x)}_{\text{хомогени део}} + g(x) \to \text{нехомогени}$$

$$\text{ОР}: Y(x) = \underbrace{Y_H(x)}_{\text{реш. хом.}} + \overbrace{Y_P(x)}^{\text{парт. реш. нехом.}}$$

(*) $g(x) = P_s[x]\cdot e^{\mu x}$, $\mu\in\mathbb{R}$

$s\in\mathbb{N}_0$, $s = st(P_s[x]) = deg(P_s[x])$ $\Rightarrow Y_P(x) = Q_{m+s}[x]\cdot e^{\mu x}$

m = вишеструкост броја μ као решења карakt. јне A

1) а)
$$y_1' = 2y_1+y_2+xe^x$$
$$y_2' = -y_1+2y_2-e^x$$

б)
$$y_1' = 2y_1+y_2+2e^x$$
$$y_2' = y_1+2y_2-3e^{4x}$$

а) $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ $\Rightarrow Y' = AY + g(x)$

$$g(x) = \begin{bmatrix} xe^x \\ -2e^x \end{bmatrix} = \begin{bmatrix} x \\ -1 \end{bmatrix}\cdot e^x$$

хомогена: $Y' = AY$

$\det(A-\lambda E) = 0 \Rightarrow \lambda_{1/2} = 2\pm i$

$\lambda_1 = 2+i \Rightarrow (A-\lambda_1 E)\gamma_1 = 0 \Rightarrow \gamma_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$$\varphi(x) = \gamma_1\cdot e^{\lambda_1 x} = e^{2x}\cdot\begin{bmatrix} \cos x + i\sin x \\ i\cos x - \sin x \end{bmatrix} \begin{matrix} \to Re \\ \to Im \end{matrix}$$

$$\text{ОРХ}: Y_H(x) = c_1 e^{2x}\cdot\begin{bmatrix} \cos x \\ -\sin x \end{bmatrix} + c_2\cdot e^{2x}\cdot\begin{bmatrix} \sin x \\ \cos x \end{bmatrix}, \quad c_1, c_2\in\mathbb{R}$$

ПАРТ: $g(x) = \begin{bmatrix} x \\ -1 \end{bmatrix}\cdot e^x = P_s[x]\cdot e^{\mu x}$

$\mu = 1$ $\xrightarrow{\lambda_{1/2}=2\pm i}$ μ није сопс. вр. од $A \Rightarrow m = 0$

$s = 1$

$$Y_P(x) = Q_{s+m}[x]\cdot e^{\mu x} = Q_1[x]\cdot e^x = \begin{bmatrix} a_1x+b_1 \\ a_2x+b_2 \end{bmatrix}\cdot e^x \Rightarrow Y_P' = AY_P + g$$

$$\cancel{e^x}\cdot\begin{bmatrix} a_1x+b_1+a_1 \\ a_2x+b_2+a_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\cdot\begin{bmatrix} a_1x+b_1 \\ a_2x+b_2 \end{bmatrix}\cancel{e^x} + \begin{bmatrix} x \\ -1 \end{bmatrix}\cdot\cancel{e^x}$$

$$\cancel{e}\cdot\begin{bmatrix} a_2x+b_2+a_2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \end{bmatrix}\cdot\begin{bmatrix} a_2x+b_2 \end{bmatrix}\cancel{e} + \begin{bmatrix} -1 \end{bmatrix}\cdot\cancel{e}$$

$$\left.\begin{aligned} a_1x+a_1+b_1 &= (2a_1+a_2+1)x + 2b_1+b_2 \\ a_2x+a_2+b_2 &= (-a_1+2a_2)x - b_1+2b_2-1 \end{aligned}\right\} \forall x\in\mathbb{R} \Rightarrow$$

$a_1 = 2a_1+a_2+1 \Rightarrow a_1=a_2=-\frac{1}{2}$

$a_1+b_1 = 2b_1+b_2$

$a_2 = -a_1+2a_2 \rightsquigarrow a_1=a_2$

$a_2+b_2 = -b_1+2b_2-1$

$$\left.\begin{aligned} b_1+b_2 &= -\frac{1}{2} \\ -b_1+b_2 &= -\frac{1}{2}+1=\frac{1}{2} \end{aligned}\right\} \Rightarrow b_1=-\frac{1}{2},\quad b_2=0$$

$$y_p(x) = \begin{bmatrix} -\frac{1}{2}x-\frac{1}{2} \\ -\frac{1}{2}x \end{bmatrix}\cdot e^x$$

ОР: $y(x) = y_H(x)+y_p(x)$.

б) $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\qquad y' = Ay+g$

$g(x) = \begin{bmatrix} 2e^{x} \\ -3e^{4x} \end{bmatrix} \rightsquigarrow g(x) = P_s[x]\cdot e^{\mu x}$ не можемо!

$g(x) = g_1(x)+g_2(x)$

$g_1(x) = \begin{bmatrix} 2e^x \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}\cdot e^x$, $g_2(x) = \begin{bmatrix} 0 \\ -3e^{4x} \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}\cdot e^{4x}$

$y_p(x) = y_{p_1}(x) + y_{p_2}(x)$

$\lambda_1=1,\ \lambda_2=3$

$\gamma_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\qquad \gamma_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\varphi_1(x) = \begin{bmatrix} e^x \\ -e^x \end{bmatrix}$ $\qquad \varphi_2(x) = \begin{bmatrix} e^{3x} \\ e^{3x} \end{bmatrix}$

$\underline{g_1(x)}$: $\quad g_1(x) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}\cdot e^x - P_s[x]\cdot e^{\mu x}$

$$\left.\begin{aligned} s&=0 \\ \mu&=1 \Rightarrow m=1 \\ &(\lambda_1=1) \end{aligned}\right\} \Rightarrow y_{p_1}(x) = Q_1[x]\cdot e^x = \begin{bmatrix} a_1x+b_1 \\ a_2x+b_2 \end{bmatrix}\cdot e^x$$

$$\cancel{e^x}\cdot\begin{bmatrix} a_1x+b_1+a_1 \\ a_2x+b_2+a_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\cdot\begin{bmatrix} a_1x+b_1 \\ a_2x+b_2 \end{bmatrix}\cancel{e^x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}\cdot\cancel{e^x}$$

$a_1 = 2a_1+a_2$

$a_1+b_1 = 2b_1+b_2+2$

$a_2 = a_1+2a_2$

$0 = a_1+a_2$

$a_1 = b_1+b_2+2$

$-a_1 = b_1+b_2$

$a_1 = -a_1+2$

$a_1 = 1 \Rightarrow a_2=-1$

$$a_1+b_1 = 2b_1+b_2+2 \qquad a_1=b_1+b_2+2$$
$$0=a_1+a_2 \qquad a_2 = a_1+2a_2 \qquad -a_1=b_1+b_2$$
$$a_2=-a_1 \qquad a_2+b_2=b_1+2b_2$$

$a_1=-a_1+2$

$a_1=1 \Rightarrow a_2=-1$

$b_1+b_2=-1$

нпр. $b_1=-1$

$b_2=0$

$$y_{p_1}(x)=\begin{bmatrix} x-1 \\ -x \end{bmatrix}\cdot e^x$$

$\underline{\underline{g_2(x)}}$:

$$g_2(x)=\begin{bmatrix} 0 \\ -3 \end{bmatrix}\cdot e^{4x} \Rightarrow \left.\begin{matrix} s=0 \\ \mu=4 \Rightarrow m=0 \end{matrix}\right\} \Rightarrow y_{p_2}(x)=Q_0[x]\cdot e^{4x}=\begin{bmatrix} a \\ b \end{bmatrix}\cdot e^{4x}$$

$$y_{p_2}'=Ay_{p_2}+g_2$$

$$\begin{bmatrix} a \\ b \end{bmatrix}\cdot 4\cdot \cancel{e^{4x}}=\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\cdot\begin{bmatrix} a \\ b \end{bmatrix}\cdot\cancel{e^{4x}}+\begin{bmatrix} 0 \\ -3 \end{bmatrix}\cdot\cancel{e^{4x}}$$

$$\left.\begin{matrix} 4a=2a+b \\ 4b=a+2b-3 \end{matrix}\right\} \begin{matrix} b=2a \\ 8a=a+4a-3 \Rightarrow a=-1 \Rightarrow b=-2 \end{matrix}$$

$$y_{p_2}(x)=\begin{bmatrix} -1 \\ -2 \end{bmatrix}\cdot e^{4x}$$

$$\text{OP: } y(x)=c_1\cdot\begin{bmatrix} e^x \\ -e^x \end{bmatrix}+c_2\cdot\begin{bmatrix} e^{3x} \\ e^{3x} \end{bmatrix}+\begin{bmatrix} x-1 \\ -x \end{bmatrix}\cdot e^x+\begin{bmatrix} -1 \\ -2 \end{bmatrix}\cdot e^{4x}, \quad c_1,c_2\in\mathbb{R}.$$

(2)
$$y_1'=y_2+\operatorname{tg}^2x+1 \quad ?$$
$$y_2'=-y_1+\operatorname{tg}x$$

$$y=\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$A=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$g(x)=\begin{bmatrix} \operatorname{tg}^2x+1 \\ \operatorname{tg}x \end{bmatrix}=\begin{bmatrix} \frac{\sin^2x}{\cos^2}+\frac{\cos^2x}{\cos^2x} \\ \operatorname{tg}x \end{bmatrix}=\begin{bmatrix} \frac{1}{\cos^2x} \\ \operatorname{tg}x \end{bmatrix}$$

$y' = A(x)\cdot y$

$\text{OP: } y(x)=\underbrace{\Phi(x)}\cdot c, \quad c\in\mathbb{R}^n$

↳ фундаментална матрица

$\Phi(x)=[y_1{\downarrow} \;\; y_2{\downarrow} \;\; \cdots \;\; y_n{\downarrow}]$

$y'=A(x)\cdot y+g(x)$

$y(x)=\Phi(x)\cdot\left(c+\int\Phi^{-1}(x)\cdot g(x)\,dx\right)$

$=\underbrace{\boxed{\Phi(x)\cdot c}}_{y_H(x)}+\underbrace{\boxed{\Phi(x)\cdot\int\Phi^{-1}(x)\,g(x)\,dx}}_{y_P(x)}$

$y_H? \rightsquigarrow \Phi(x)$

$$\left.\begin{matrix} y_1'=y_2 \\ \end{matrix}\right\} \; y_1''=y_2'=-y_1 \Rightarrow y_1''+y_1=0 \rightsquigarrow y_1=c_1\cos x+c_2\sin x$$

$$\left.\begin{array}{l} y_1' = y_2 \\ y_2' = -y_1 \end{array}\right\} \quad y_1'' = y_2' = -y_1 \Rightarrow y_1'' + y_1 = 0 \rightsquigarrow \begin{array}{l} y_1 = c_1 \cos x + c_2 \sin x \\ y_2 = -c_1 \sin x + c_2 \cos x \end{array}$$

$$\varphi_1(x) = \begin{bmatrix} \cos x \\ -\sin x \end{bmatrix}, \quad \varphi_2(x) = \begin{bmatrix} \sin x \\ \cos x \end{bmatrix}$$

$$\Phi(x) = \begin{bmatrix} \varphi_1 \downarrow & \varphi_2 \downarrow \end{bmatrix} = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}, \quad W(x) = \det \Phi(x) = \cos^2 x + \sin^2 x = 1$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to A^{-1} = \frac{1}{ad-bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \Rightarrow \Phi^{-1}(x) = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$$

$$\int \Phi^{-1}(x) \cdot g(x)\, dx = \int \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\cos^2 x} \\ \operatorname{tg} x \end{bmatrix} dx =$$

$$= \int \begin{bmatrix} \frac{1}{\cos x} - \frac{\sin^2 x}{\cos x} \\ \frac{\sin x}{\cos^2 x} + \sin x \end{bmatrix} dx = \int \begin{bmatrix} \frac{\overset{\cos^2 x}{1 - \sin^2 x}}{\cos x} \\ \frac{\sin x}{\cos^2 x} + \sin x \end{bmatrix} dx = \int \begin{bmatrix} \cos x \\ \sin x + \frac{\sin x}{\cos^2 x} \end{bmatrix} dx =$$

$$= \begin{bmatrix} \int \cos x \, dx \\ \int \left(\sin x + \frac{\sin x}{\cos^2 x}\right) dx \end{bmatrix} = \begin{bmatrix} \sin x \\ -\cos x + \frac{1}{\cos x} \end{bmatrix}$$

$$\int \frac{\sin x}{\cos^2 x} dx \underset{\substack{\uparrow \\ \cos x = t \\ -\sin x\, dx = dt}}{=} \int \frac{-dt}{t^2} = -\int t^{-2} dt = -\frac{t^{-2+1}}{-2+1} + c = \frac{1}{t} + c = \frac{1}{\cos x} + c$$

$$y_P(x) = \Phi(x) \cdot \int \Phi^{-1}(x) g(x)\, dx = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} \sin x \\ \frac{1}{\cos x} - \cos x \end{bmatrix} = \begin{bmatrix} \sin x \cos x + \operatorname{tg} x - \sin x \cos x \\ -\sin^2 x + 1 - \cos^2 x \end{bmatrix} = \begin{bmatrix} \operatorname{tg} x \\ 0 \end{bmatrix}$$

$$\text{OP}: \; y(x) = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} \operatorname{tg} x \\ 0 \end{bmatrix}.$$

Експоненти матрице

$$A \in M_n(\mathbb{R}) \;(\mathbb{C}) \rightsquigarrow \exp : M_n(\mathbb{R}) \to M_n(\mathbb{R})$$

$$\exp(A) = e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

$$\left[e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}\right]$$

Особине експонента: $\quad (\forall A, B \in M_n(\mathbb{R}))$

$\boxed{(1)}$ $\frac{d}{dt}(e^{tA}) = Ae^{tA}$

$\boxed{(2)}$ $AB = BA \Rightarrow e^{A+B} = e^A \cdot e^B$

$\boxed{(3)}$ $AB = BA \Rightarrow Ae^B = e^B A$

$\boxed{(4)}$ $\det(e^A) = e^{\operatorname{tr} A}$

$\boxed{(5)}$ $\lim\limits_{n\to\infty} \left(E + \frac{A}{n}\right)^n = e^A$

③ Испитати да ли постоји реална матрица A таква да је:

а) $e^A = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$, б) $e^A = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}$

а) $e^A \in M_2(\mathbb{R}) \Rightarrow A \in M_2(\mathbb{R})$

$\Rightarrow \operatorname{tr} A \in \mathbb{R} \Rightarrow e^{\operatorname{tr} A} > 0$

$(4) \Rightarrow \det(e^A) = e^{\operatorname{tr} A} > 0$

$e^A = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix} \Rightarrow \det(e^A) = \det\left(\begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}\right) = -4 \Big\}\ \lightning \Rightarrow \nexists$

б) $\left.\begin{aligned}\det(e^A) &= \det\left(\begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}\right) = 4 > 0 \\ &= e^{\operatorname{tr} A}\end{aligned}\right\} \operatorname{tr} A = \ln(4).$

$(3) \Rightarrow AA = A^2 = AA \Rightarrow \underbrace{Ae^A = e^A A}$

$e^A = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}$

$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$

$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$

$\cancel{-\alpha = -\alpha}$

$\left.\begin{aligned} -4\beta &= -\beta \\ -\gamma &= -4\gamma \end{aligned}\right\} \beta = \gamma = 0$

$\cancel{-4\delta = -4\delta}$

$$\left.\begin{aligned} -4\beta &= -\beta \\ -\gamma &= -4\gamma \end{aligned}\right\} \beta = \gamma = 0$$

$$-4\delta = -4\delta$$

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}, \quad A^2 = \begin{bmatrix} \alpha^2 & 0 \\ 0 & \delta^2 \end{bmatrix}, \quad A^3 = \begin{bmatrix} \alpha^3 & 0 \\ 0 & \delta^3 \end{bmatrix}, \ \ldots, \ A^k = \begin{bmatrix} \alpha^k & 0 \\ 0 & \delta^k \end{bmatrix}$$ (формално индукцијом)

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \frac{\begin{bmatrix} \alpha^k & 0 \\ 0 & \delta^k \end{bmatrix}}{k!} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{\delta^k}{k!} \end{bmatrix} = \begin{bmatrix} e^\alpha & 0 \\ 0 & e^\delta \end{bmatrix}$$

$$e^A = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} e^\alpha & 0 \\ 0 & e^\delta \end{bmatrix} \Rightarrow \left.\begin{aligned} e^\alpha &= -1 \\ e^\delta &= -4 \end{aligned}\right\} \lightning \Rightarrow \nexists$$

④ Нека су $A, B \in M_n(\mathbb{R})$ и нека $\exists B^{-1}$. Доказати $e^{B^{-1}AB} = B^{-1} e^A B$.

$$e^{B^{-1}AB} = \sum_{k=0}^{\infty} \frac{(B^{-1}AB)^k}{k!} = \sum_{k=0}^{\infty} \frac{B^{-1}A^k B}{k!} = B^{-1} \cdot \left(\sum_{k=0}^{\infty} \frac{A^k}{k!} \right) \cdot B = B^{-1} e^A B.$$

$$(B^{-1}AB)^k = \overset{1.}{(B^{-1}AB)}\overset{2.}{(B^{-1}AB)}\overset{3.}{(B^{-1}AB)} \ldots \overset{k.}{(B^{-1}AB)} =$$

$$= B^{-1}A\underbrace{(BB^{-1})}_{E}A\underbrace{(BB^{-1})}_{E}A(\underbrace{B}_{E} \ldots \underbrace{B^{-1}}_{E})AB =$$

$$= B^{-1}\overbrace{AAA\ldots A}^{k}B = B^{-1}A^kB.$$