Парцијалне ДЈ 1.реда

$u(x) \leftarrow$ ОДЈ

$u(x_1,\dots,x_n) \leftarrow$ ПДЈ

- $f\left(x_1,\dots,x_n,u,\underbrace{\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n}}_{1.\text{реда}}\right)=0$, $n\geq 2$, општи облик ПДЈ 1.реда

- Квазилинеарне ПДЈ 1.реда (КВЛ): $\sum X_k(x_1,\dots,x_n,u)\cdot\frac{\partial u}{\partial x_k}=R(x_1,\dots,x_n,u)$

  спец: хомогена (ХЛ) линеарна ПДЈ 1.реда

  $$\sum X_k(x_1,\dots,x_n)\cdot\frac{\partial u}{\partial x_k}=0$$

  $X_k : \underset{\mathbb{R}^n}{\overset{\cup\!\shortmid}{D}} \to \mathbb{R}$

- 1) КВЛ $\rightsquigarrow$ ХЛ

  $$\sum X_k(x_1,\dots,x_n,u)\cdot\frac{\partial u}{\partial x_k}=R(x_1,\dots,x_n,u)$$

  $$\Big\downarrow\ v=v(x_1,\dots,x_n,u)$$

  $$\sum X_k(x_1,\dots,x_n,u)\cdot\frac{\partial v}{\partial x_k}+R(x_1,\dots,x_n,u)\cdot\frac{\partial v}{\partial u}=0 \qquad (\Diamond)$$

- 2) ХЛ $\rightsquigarrow$ систем карактеристика

  $$\sum X_k(x_1,\dots,x_n)\cdot\frac{\partial u}{\partial x_k}=0 \qquad (\#)$$

  $$\Big\downarrow$$

  $$\frac{dx_1}{X_1(x_1,\dots,x_n)}=\frac{dx_2}{X_2(x_1,\dots,x_n)}=\dots=\frac{dx_n}{X_n(x_1,\dots,x_n)} \qquad (*)$$

- 3) Нађи $\underset{\text{независне}}{n}$ прве интеграле $(*)$: $\psi_1,\dots,\psi_{n-1}$ $\Rightarrow$ ОР од $(\#)$: $u=\varphi(\psi_1,\dots,\psi_{n-1})$, $\varphi\in C^1(\mathbb{R}^{n-1})$.

  ОР од $(\Diamond)$: $\varphi(\psi_1,\dots,\psi_n)=0$, $\varphi\in C^1(\mathbb{R}^n)$.

(1) $m,n,k\in\mathbb{R}\setminus\{0\}$. Решити ПДЈ:

$$(mz-ny)\cdot\frac{\partial u}{\partial x}+(nx-kz)\cdot\frac{\partial u}{\partial y}+(ky-mx)\cdot\frac{\partial u}{\partial z}=0 \rightsquigarrow \text{ХЛ}$$

$$\Big\downarrow$$

$$\frac{dx}{mz-ny} = \frac{dy}{nx-kz} = \frac{dz}{ky-mx} \qquad \psi_1, \psi_2 = ?$$

$$\frac{\alpha\,dx+\beta\,dy}{\alpha(mz-ny)+\beta(nx-kz)} = \frac{dz}{ky-mx}$$

уз z: $\alpha m - \beta k = 0$

$\alpha = k,\ \beta = m \Rightarrow$

$$\frac{k\,dx+m\,dy}{n\cdot mx - nk\cdot y} = \frac{dz}{\cancel{ky-mx}} \Rightarrow k\,dx + m\,dy + n\,dz = 0 \ / \int$$

$-n$

$$kx+my+nz = C_1$$

$$\psi_1(x,y,z) = kx+my+nz$$

$$\frac{\alpha x\,dx+\beta y\,dy}{\alpha x(mz-ny)+\beta y(nx-kz)} = \frac{z\,dz}{kyz-mxz}$$

$\vdots$

$\alpha = \beta = -1$: $\quad -x\,dx - y\,dy = z\,dz \ / \int$

$$x^2+y^2+z^2 = \psi_2(x,y,z)$$

ψ_1, ψ_2 нез?
$$\frac{D(\psi_1,\psi_2)}{D(x,y)} = \begin{vmatrix} k & m \\ 2x & 2y \end{vmatrix} = 2(ky-mx) \neq 0$$

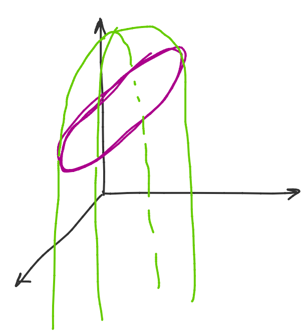
нез. пром. z

ОР: $u = \underline{\varphi}(kx+my+nz,\ x^2+y^2+z^2)$, $\varphi \in C^1(\mathbb{R}^2)$

- Кошијев проблем: Одредити решење које задовољава $u\big|_{x_k=t} = f(x_1,\dots,\hat{x}_k,\dots,x_n)$ — не појављује се x_k

→ наћи решење које садржи дату „површ"

$\varphi = ?$



$$\overline{\psi}_l = \psi_l(x_1,\dots,x_{k-1},t,x_{k+1},\dots,x_n)$$

$u\big|_{x_k=t} = f = \underline{g}(\overline{\psi}_1,\dots,\overline{\psi}_{n-1})$. ← наћи овакво g

$\Rightarrow$ Кошијево решење: $u = g(\psi_1,\dots,\psi_{n-1})$.

Ово је за ХЛ. За КВЛ иде аналогно.

(2) Решити Кошијев проблем

$$x(z^2-y^2)\cdot\frac{\partial u}{\partial x}+y(x^2+z^2)\frac{\partial u}{\partial y}-z(x^2+y^2)\cdot\frac{\partial u}{\partial z}=0,\quad u\big|_{x=1}=u(1,y,z)=(y+z)^2.$$

$$\text{ХЛ} \rightsquigarrow \frac{dx}{x(z^2-y^2)}=\frac{dy}{y(x^2+z^2)}=\frac{dz}{-z(x^2+y^2)}$$

(2) са прошлог двочаса

$$\left.\begin{aligned}\psi_1&=x^2+y^2+z^2\\ \psi_2&=\frac{x}{yz}\end{aligned}\right\}\ \text{ОР}:\ u=\varphi\left(x^2+y^2+z^2,\frac{x}{yz}\right),\ \varphi\in C^1(\mathbb{R}^2).$$

Тражимо φ тд: $u(1,y,z)=\varphi\left(1+y^2+z^2,\frac{1}{yz}\right)=(y+z)^2$?

$$\overline{\psi}_1=\psi_1(1,y,z)=1+y^2+z^2$$

$$\overline{\psi}_2=\psi_2(1,y,z)=\frac{1}{yz}$$

$$g(\overline{\psi}_1,\overline{\psi}_2)=f \Rightarrow g\left(1+y^2+z^2,\frac{1}{yz}\right)=(y+z)^2,\ g=?$$

$$(y+z)^2=y^2+z^2+2yz=\underbrace{(y^2+z^2+1)}_{=\overline{\psi}_1}-1+\frac{2}{\underbrace{\frac{1}{yz}}_{=\overline{\psi}_2}}=\overline{\psi}_1-1+\frac{2}{\overline{\psi}_2}$$

$$g(\overline{\psi}_1,\overline{\psi}_2)=\overline{\psi}_1-1+\frac{2}{\overline{\psi}_2}$$

Кошијево решење: $u=g(\psi_1,\psi_2)=\psi_1-1+\frac{2}{\psi_2}=x^2+y^2+z^2-1+\frac{2yz}{x}$.

(3) Решити Кошијев проблем

$$x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}+z^2=0,\qquad z=1,\ xy=x+y$$

$$x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=\underbrace{-z^2}_{=R(x,y,z)}$$

КВЛ $\rightsquigarrow u(x,y,z)$

$$\text{ХЛ}:\ x^2\cdot\frac{\partial u}{\partial x}+y^2\cdot\frac{\partial u}{\partial y}-z^2\cdot\frac{\partial u}{\partial z}=0$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{-z^2}$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} \,\Big/ \int$$

$$-\frac{1}{x} = -\frac{1}{y} + C_1$$

$$\psi_1 = \frac{1}{x} - \frac{1}{y}$$

$$\frac{dy}{y^2} = -\frac{dz}{z^2} \,\Big/ \int$$

$$-\frac{1}{y} = \frac{1}{z} + C_2$$

$$\psi_2 = \frac{1}{y} + \frac{1}{z}$$

$$\frac{D(\psi_1,\psi_2)}{D(x,z)} = \begin{vmatrix} -\frac{1}{x^2} & 0 \\ 0 & -\frac{1}{z^2} \end{vmatrix} = \frac{1}{x^2 z^2} \neq 0$$

није се тражило

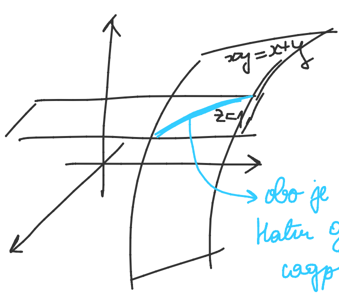
ОР ХЛ: $u = \varphi\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{y} + \frac{1}{z}\right)$

ОР КВЛ: $\varphi\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{y} + \frac{1}{z}\right) = 0$ ← имплицитна ф-ја $z(x,y)$

$\varphi \in C^1(\mathbb{R}^2)$

Кошијев проблем: $z = 1$

$xy = x + y$



ово је пресек

Наћи g, тд. решење садржи ову криву

$$\left.\begin{aligned} \overline{\psi}_1 &= \frac{1}{x} - \frac{1}{y} \\ \overline{\psi}_2 &= \frac{1}{y} + 1 \end{aligned}\right\}$$ треба да их доведемо у везу при услову $xy = x + y$

$$xy = x + y \;/ : xy$$

$$1 = \frac{1}{y} + \frac{1}{x} = \underbrace{\left(\frac{1}{x} - \frac{1}{y}\right)}_{\overline{\psi}_1} + 2\underbrace{\left(\frac{1}{y} + 1\right)}_{\overline{\psi}_2} - 2$$

$$1 = \overline{\psi}_1 + 2\overline{\psi}_2 - 2$$

$$\Rightarrow \overline{\psi}_1 + 2\overline{\psi}_2 - 3 = 0$$

g

$$g(\overline{\psi}_1, \overline{\psi}_2) = \overline{\psi}_1 + 2\overline{\psi}_2 - 3$$

Кошијево решење:

$$0 = g(\psi_1, \psi_2) = \psi_1 + 2\psi_2 - 3$$

$$\frac{1}{x} - \frac{1}{y} + 2\left(\frac{1}{y} + \frac{1}{z}\right) - 3 = 0$$

имплицитно задато по $z(x,y)$ → $\frac{1}{x} + \frac{1}{y} + \frac{2}{z} = 3$

Може чак и експлицитно: $\frac{2}{z} = 3 - \frac{1}{x} - \frac{1}{y}$

$$z = \frac{2}{3 - \left(\frac{1}{x} + \frac{1}{y}\right)}.$$

II начин за Кошијев: параметризација криве $z=1$

$$xy = x+y$$
$$y(x-1) = x$$
$$y = \frac{x}{x-1}$$

$$\gamma(t) = \left(t, \frac{t}{t-1}, 1\right)$$

$$\overline{\psi_1} = \psi_1\Big|_{\gamma} = \psi_1\left(t, \frac{t}{t-1}, 1\right) = \frac{2}{t} - 1 \qquad \frac{1}{t} - \frac{t-1}{t} = \frac{1-t+1}{t} = \frac{2}{t} - 1$$

$$\overline{\psi_2} = \psi_2\Big|_{\gamma} = \psi_2\left(t, \frac{t}{t-1}, 1\right) = 2 - \frac{1}{t} \qquad \frac{t-1}{t} + 1 = \frac{2t-1}{t} = 2 - \frac{1}{t}$$

$$\overline{\psi_1} + 1 = 2(2 - \overline{\psi_2}) \Rightarrow \overline{\psi_1} + 2\overline{\psi_2} - 3 = 0 \ldots$$

④ Решити Кошијев проблем

$$x(x^2+3y^2)\frac{\partial z}{\partial x} + y(3x^2+y^2)\frac{\partial z}{\partial y} = 2z(x^2+y^2), \qquad \begin{matrix} xy = z \\ x^2 - y^2 = z^2 \end{matrix}$$

КВЛ $\longrightarrow u(x,y,z)$

ХЛ: $x(x^2+3y^2)\frac{\partial u}{\partial x} + y(3x^2+y^2)\frac{\partial u}{\partial y} + 2z(x^2+y^2)\frac{\partial u}{\partial z} = 0$

$$\frac{dx}{x(x^2+3y^2)} = \frac{dy}{y(3x^2+y^2)} = \frac{dz}{2z(x^2+y^2)} \qquad \psi_1, \psi_2 = ?$$

$$\frac{\frac{\alpha}{x}dx + \frac{\beta}{y}dy}{\alpha(x^2+3y^2) + \beta(3x^2+y^2)} = \frac{\frac{dz}{z}}{2(x^2+y^2)}$$

$$x^2(\alpha + 3\beta) + y^2(3\alpha + \beta)$$

хоћемо: $\left.\begin{matrix} \alpha + 3\beta = 2 \\ 3\alpha + \beta = 2 \end{matrix}\right\} \quad \alpha = \beta = \frac{1}{2}$

$$\frac{1}{2}\cdot\frac{dx}{x} + \frac{1}{2}\cdot\frac{dy}{y} = \frac{dz}{z} \quad / \int$$

$$\frac{1}{2}\cdot\frac{dx}{x}+\frac{1}{2}\cdot\frac{dy}{y}=\frac{dz}{z}\ \Big/\int$$

$$\frac{1}{2}\ln|x|+\frac{1}{2}\ln|y|=\ln|z|+c_1 \ \ldots \quad \psi_1(x,y,z)=\frac{xy}{z^2}$$

$$\frac{\alpha x\,dx+\beta y\,dy}{\alpha x^2(x^2+3y^2)+\beta y^2(3x^2+y^2)}=\frac{\frac{dz}{z}}{2(x^2+y^2)}$$

свакако делимо са z, јер се нигде друго не појављује

$$\alpha x^4+\beta y^4+x^2y^2(3\alpha+3\beta)$$

идеја: 1) направити квадрат бинома ✗

2) направити $x^4-y^4=(x^2-y^2)(x^2+y^2)$

$\alpha=1,\ \beta=-1$

$$\frac{x\,dx-y\,dy}{x^4-y^4+\cancel{x^2y^2(0)}}=\frac{\frac{dz}{z}}{2(x^2+y^2)}$$

$$\frac{x\,dx-y\,dy}{\cancel{(x^2+y^2)}(x^2-y^2)}=\frac{\frac{dz}{z}}{2\cancel{(x^2+y^2)}}\Rightarrow\frac{2x\,dx-2y\,dy}{\boxed{x^2-y^2}}=\frac{dz}{z}\Rightarrow d(\ln|x^2-y^2|)=d(\ln|z|)\ \Big/\int$$

$$d(\ln|x^2-y^2|)=\frac{1}{x^2-y^2}\cdot 2x\,dx+\frac{1}{x^2-y^2}\cdot(-2y)\,dy=\frac{2x\,dx-2y\,dy}{x^2-y^2}$$

$$\ln|x^2-y^2|=|z|+C_2$$

$\vdots$

$$\psi_2=\frac{x^2-y^2}{z}$$

ψ_1,ψ_2 – нез ← доказати

Кошијево реш? $\left.\begin{array}{l} xy=z \\ x^2-y^2=z^2 \end{array}\right\}$ у пресеку је крива c

$$\overline{\psi_1}=\psi_1\big|_c=\left(\frac{xy}{z^2}\right)\Big|_c\underset{xy=z}{=}\frac{z}{z^2}=\frac{1}{z}$$

циљ: добити оба преко истог параметра!

$$\overline{\psi_2}=\psi_2\big|_c=\left(\frac{x^2-y^2}{z}\right)\Big|_c=\frac{z^2}{z}=z$$

$$\overline{\psi}_2 = \psi_2\Big|_C = \left(\frac{x^2-y^2}{z}\right)\Bigg|_C = \frac{z^2}{z} = z$$

$$x^2-y^2=z^2$$

параметра!

$$\overline{\psi}_1 = \frac{1}{\overline{\psi}_2} \Rightarrow \overline{\psi}_1 - \frac{1}{\overline{\psi}_2} = 0$$

$$g(\overline{\psi}_1, \overline{\psi}_2) = \overline{\psi}_1 - \frac{1}{\overline{\psi}_2}$$

КР: $g(\psi_1, \psi_2) = 0$

$$\psi_1 - \frac{1}{\psi_2} = 0$$

$$\frac{xy}{z^2} - \frac{z}{x^2-y^2} = 0 \Rightarrow xy(x^2-y^2) = z^3.$$