1 - уторак 23.02.2021.

① Свести систем ДЈ на систем ДЈ у нормалном облику

$$y''' = xy'' + y - z' + x + 1 \rightarrow 3.\ \text{реда}$$
$$z'' = y' \sin x + y - z + x^2 \rightarrow 2.\ \text{реда}$$

5 јна 1. реда

$$y_1' = f_1(x, y_1, \dots, y_n)$$
$$y_2' = f_2(x, y_1, \dots, y_n)$$
$$\vdots$$
$$y_n' = f_n(x, y_1, \dots, y_n)$$

1. реда

y

$y_1 = y'$

$y_2 = y_1' = y''$

$y''' = y_2'$

z

$z_1 = z'$

$z'' = z_1'$

(y, y_1, y_2, z, z_1)

$$\left.\begin{aligned} y_2' &= x \cdot y_2 + y - z_1 + x + 1 \\ z_1' &= y_1 \cdot \sin x + y - z + x^2 \\ y' &= y_1 \\ y_1' &= y_2 \\ z' &= z_1 \end{aligned}\right\} \text{у норм. облику}$$

② Методом елиминације решити систем ДЈ:

а) $y' = py - qz$

$z' = qy + pz$

$p, q \in \mathbb{R} \setminus \{0\}$

б) $y_1' = y_2$

$y_2' = y_1$

$y_3' = y_1 + y_2 + y_3$

а) $y' = py - qz \Rightarrow z = \dfrac{py - y'}{q} \Rightarrow z' = \dfrac{py' - y''}{q}$

$z' = qy + pz \Rightarrow \dfrac{py' - y''}{q} = qy + p \cdot \dfrac{py - y'}{q} \ / \cdot q$

$py' - y'' = q^2 y + p^2 y - py'$

$y'' - 2py' + (p^2 + q^2)y = 0$

$\lambda^2 - 2p\lambda + (p^2 + q^2) = 0$

$D = 4p^2 - 4(p^2 + q^2) = -4q^2 < 0$

$$D = 4p^2 - 4(p^2+q^2) = -4q^2 < 0$$

$$y = c_1 \cdot e^{px}\cos(qx) + c_2 e^{px}\sin(qx)$$

$c_1, c_2 \in \mathbb{R}$

$$\lambda_{1/2} = \frac{2p \pm i \cdot 2q}{2} = p \pm iq$$

$$y' = c_1 e^{px}\left(-\sin(qx)\cdot q + p\cdot\cos(qx)\right) + c_2 e^{px}\left(\cos(qx)\cdot q + p\cdot\sin(qx)\right)$$

$$z = \frac{p}{q}y - \frac{1}{q}\cdot y' = -c_2 e^{px}\cos(qx) + c_1 e^{px}\sin(qx)$$

б) $y_1' = y_2$

$y_2' = y_1 \Rightarrow y_1' = y_2''$

$\Rightarrow y_2'' = y_2 \Rightarrow \dots$ ($\lambda^2 - 1 = 0$)

$c_1, c_2 \in \mathbb{R}$

$$\begin{cases} y_2 = c_1 e^{x} + c_2 e^{-x} \\ y_1 = y_2' = c_1 e^{x} - c_2 e^{-x} \end{cases}$$

$$y_3' - y_3 = y_1 + y_2 = 2c_1 e^{x}$$

$p(x) = -1$

$q(x) = 2c_1 e^{x}$

$\int p(x)\,dx = -x$

$\int q(x)\cdot e^{\int p(x)dx}\,dx = \int 2c_1 e^{x}\cdot e^{-x}\,dx = \int 2c_1\,dx = 2c_1 x$

$$\boxed{y_3 = e^{x}\cdot(c_3 + 2c_1 x)}, \quad c_3 \in \mathbb{R}$$

③ Помоћу одговарајуће смене независне променљиве решити систем ДЈ:

а) $2\sqrt{x}\cdot y' = 2y - z$

$2\sqrt{x}\cdot z' = y + 2z$

б) $xy_1' = y_1 + y_2$

$xy_2' = y_2$

$xy_3' = -y_3$

$xy_2' = y_2$

$x\frac{dy_2}{dx} = y_2$

$\frac{dy_2}{y_2} = \frac{dx}{x}$

$y_2 = cx$

$y_3 = \dots$

$y_1 = \dots$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad y' = F(x, y)$$

$f(x)\cdot y' = F(x, y)$

$f(x) = 2\sqrt{x}, x, \dots$

$x \rightsquigarrow t$

нека је: $f(x)\cdot\frac{dy}{dx} = \frac{dy}{dt}$

$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$

$f(x)\cdot\cancel{\frac{dy}{dx}} = \cancel{\frac{dy}{dx}}\cdot\frac{dx}{dt}$

$\frac{dx}{dt} = f(x)$

$$\boxed{\frac{dt}{dx} = \frac{1}{f(x)}}$$

$$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$$

$$\boxed{\frac{dt}{dx} = \frac{1}{f(x)}}$$

2) $t(x) = ?$ $\quad , x \geq 0$

$$\frac{dt}{dx} = \frac{1}{f(x)} = \frac{1}{2\sqrt{x}} \quad / \int$$

$$t(x) = \sqrt{x} \Rightarrow x = t^2$$

$$\left.\begin{aligned} \frac{dy}{dt} &= \frac{dy}{dx}\cdot\frac{dx}{dt} = \frac{y'}{\frac{dt}{dx}} = \frac{y'}{\frac{1}{2\sqrt{x}}} = 2\sqrt{x}\cdot y' \\ \frac{dz}{dt} &= \ldots \quad \ldots \quad \ldots = 2\sqrt{x}\cdot z' \end{aligned}\right\}$$

$$\frac{dy}{dt} = y'_t = 2y - z \qquad \Rightarrow z = 2y - y'_t$$

$$z'_t = 2y'_t - y''_t$$

$$\frac{dz}{dt} = z'_t = y + 2z$$

$$\Downarrow$$

$$2y'_t - y''_t = y + 4y - 2y'_t$$

$$y''_t - 4y'_t + 5y = 0$$

$\lambda^2 - 4\lambda + 5 = 0$

$D = 16 - 20 = -4$

$\lambda_{1/2} = \frac{4 \pm 2i}{2} = 2 \pm i$

$$y(t) = c_1 e^{2t}\sin t + c_2 e^{2t}\cos t$$

$$y'_t = c_1 e^{2t}(2\sin t + \cos t) + c_2 e^{2t}(2\cos t - \sin t)$$

$$z(t) = 2y(t) - y'_t(t) = -c_1 e^{2t}\cos t + c_2 e^{2t}\sin t$$

$c_1, c_2 \in \mathbb{R}$

$$\left.\begin{aligned} y(x) &= c_1 e^{2\sqrt{x}}\sin\sqrt{x} + c_2 e^{2\sqrt{x}}\cos\sqrt{x} \\ z(x) &= -c_1 e^{2\sqrt{x}}\cos\sqrt{x} + c_2 e^{2\sqrt{x}}\sin\sqrt{x} \end{aligned}\right\}$$

$x = 0:$ $\left.\begin{aligned} 2y(0) - z(0) &= 0 \\ y(0) + 2z(0) &= 0 \end{aligned}\right\}$ $y(0) = z(0) = 0$

6)

$$xy_1' = y_1 + y_2$$

$$xy_2' = y_2$$

$$xy_3' = -y_3$$

$x = 0:$ $\left.\begin{aligned} y_1(0) + y_2(0) &= 0 \\ y_2(0) &= 0 \\ -y_3(0) &= 0 \end{aligned}\right\}$ $y_1(0) = y_2(0) = y_3(0) = 0$

$f(x) = x$

$x \rightsquigarrow t$

$$\frac{dt}{dx} = \frac{1}{x} \quad / \int$$

$$t(x) = \ln|x|$$

$1^\circ\ x>0\ ,\ t(x)=\ln x \Rightarrow x=e^t \Rightarrow \frac{dx}{dt}=e^t=x$

$2^\circ\ x<0\ ,\ t(x)=\ln(-x)$

$\vdots$

$$\left.\begin{aligned}\frac{dy_1}{dt}&=\frac{dy_1}{dx}\cdot\frac{dx}{dt}=y_1'\cdot x\\ \frac{dy_2}{dt}&=\ \dots\ =y_2'\cdot x\\ \frac{dy_3}{dt}&=\ \dots\ =y_3'\cdot x\end{aligned}\right\}\quad \begin{aligned}y_{1\,t}'&=y_1+y_2 \Rightarrow y_{1\,t}'-y_1=c_1e^t \Rightarrow y_1=e^t(c_3+c_1t)\\ y_{2\,t}'&=y_2 \Rightarrow y_2=c_1e^t\\ y_{3\,t}'&=-y_3 \Rightarrow y_3=c_2e^{-t}\end{aligned}$$

$$y_1=x(c_3+c_1\ln x)$$
$$y_2=c_1x$$
$$y_3=\frac{c_2}{x}$$

4) Методом елиминације решити систем ДЈ:

а) $y'=1-\frac{1}{z}$, $z'=\frac{1}{y-x}$

б) $2zy'=y^2-z^2+1$, $z'=y+z$

а) $\frac{1}{z'}=y-x\ /'$ $\qquad \frac{d}{dt}\left(\frac{1}{z'}\right)=-\frac{1}{(z')^2}\cdot\frac{d}{dx}(z')=-\frac{z''}{(z')^2}$ $\qquad$ дефинисаност: $z\neq 0$, $y\neq x$

$$-\frac{z''}{(z')^2}=y'-1 \Rightarrow y'=1-\frac{z''}{(z')^2} \Rightarrow 1-\frac{z''}{(z')^2}=1-\frac{1}{z}$$

$$zz''=(z')^2\ /:z^2 \leftarrow \text{нелинеарна ДЈ 2. реда (нема } x)$$

$(zz')'=z'\cdot z'+z\cdot z''=(z')^2+zz''$

$\left(\frac{z'}{z}\right)'=\frac{z''\cdot z-(z')^2}{z^2}$

$$\frac{z''z-(z')^2}{z^2}=0$$

$$\left(\frac{z'}{z}\right)'=0\ /\int$$

$$\frac{z'}{z}=c_1\ /\int$$

$$\boxed{\begin{aligned}z&=c_2e^{c_1x}\\ y&=x+\frac{1}{c_1c_2e^{c_1x}}\end{aligned}} \Rightarrow z'=c_1c_2e^{c_1x}$$

$c_1,c_2\neq 0$

$(z\neq 0, y\neq x)$

б) $z'=y+z \Rightarrow \big|\ y=z'-z$

б)

$$z' = y + z \Rightarrow \begin{cases} y = z' - z \\ y' = z'' - z' \end{cases}$$

$$2zy' = y^2 - z^2 + 1 \Rightarrow 2z(z'' - z') = (z' - z)^2 - z^2 + 1$$

$$2zz'' - \cancel{2zz'} = (z')^2 - \cancel{2zz'} + \cancel{z^2} - \cancel{z^2} + 1$$

$$2zz'' = (z')^2 + 1$$ ← нелин. 2. реда (нема x)

СМЕНА: $u(z) = z'$

$$u'(z) = \frac{du}{dz} = \frac{d(z')}{dz} = \frac{d(z')}{dx} \cdot \frac{dx}{dz} = z'' \cdot \frac{dx}{dz}$$

$$\Rightarrow z'' = u'(z) \cdot \frac{dz}{dx} = u' \cdot \underbrace{z'}_{u} = u'u$$

$$2z \cdot u'u = u^2 + 1$$

$$\frac{u'u}{u^2+1} = \frac{1}{2z}$$

$$\frac{2u\,du}{u^2+1} = \frac{dz}{z} \Big/ \int$$

$$\ln(u^2+1) = \ln|z| + \tilde{c}_1 \quad , \tilde{c}_1 \in \mathbb{R}$$

$$u^2 + 1 = c_1 \cdot z \quad , c_1 \in \mathbb{R} \setminus \{0\}$$

$$(z')^2 + 1 = c_1 \cdot z$$

$$z' = \pm\sqrt{c_1 z - 1} \qquad (c_1 z - 1 \geq 0)$$

1° $z' = \sqrt{c_1 z - 1}$

$$\frac{dz}{\sqrt{c_1 z - 1}} = dx \Big/ \int$$

$$\frac{2}{c_1}\sqrt{c_1 z - 1} = x + c_2 \Rightarrow c_1 z - 1 = \frac{c_1^2}{4}(x + c_2)^2 \Rightarrow z = \frac{1}{c_1}\left(1 + \frac{c_1^2}{4}(x + c_2)^2\right) \Rightarrow z' = \frac{c_1}{2}(x + c_2)$$

$$\left(\frac{2}{c_1}\sqrt{c_1 z - 1}\right)' = \frac{2}{c_1} \cdot \frac{1}{2\sqrt{c_1 z - 1}} \cdot c_1 = \frac{1}{\sqrt{c_1 z - 1}}$$

$$y = z' - z = \frac{c_1}{2}(x + c_2) - \frac{1}{c_1} - \frac{c_1}{4}(x + c_2)^2$$

2° $z' = -\sqrt{c_1 z - 1}$

$\vdots$