8 - петак 23.04.2021.

г) $A = \begin{bmatrix} -2 & -5 \\ 2 & 2 \end{bmatrix}$; д) $A = \begin{bmatrix} -1 & -1 \\ 2 & -3 \end{bmatrix}$; ђ) $A = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$.

г) $\lambda_{1/2} = \pm i\sqrt{6}$ – центар $X^* = (0,0)$

$$(A - i\sqrt{6}E) \cdot \gamma = 0 \quad \therefore \quad \gamma = \begin{bmatrix} 5 \\ -2 - i\sqrt{6} \end{bmatrix}$$

$$\Psi(t) = e^{i\sqrt{6}t} \cdot \gamma = \left(\cos(\sqrt{6}t) + i\sin(\sqrt{6}t)\right) \cdot \begin{bmatrix} 5 \\ -2 - i\sqrt{6} \end{bmatrix} \begin{matrix} \rightsquigarrow \operatorname{Re}\Psi \\ \rightsquigarrow \operatorname{Im}\Psi \end{matrix}$$

$$X(t) = c_1 \cdot \begin{bmatrix} 5\cos(\sqrt{6}t) \\ -2\cos(\sqrt{6}t) + \sqrt{6}\sin(\sqrt{6}t) \end{bmatrix} + c_2 \cdot \begin{bmatrix} 5\sin(\sqrt{6}t) \\ -2\sin(\sqrt{6}t) - \sqrt{6}\cos(\sqrt{6}t) \end{bmatrix}$$ → тешко цртати

$$T^{-1} \backslash \; A = T \cdot \underbrace{\begin{bmatrix} 0 & \sqrt{6} \\ -\sqrt{6} & 0 \end{bmatrix}}_{= J} \cdot T^{-1}, \qquad T^{-1} \backslash \; X' = AX, \qquad y = \underline{T^{-1} \cdot X}, \quad y' = T^{-1} \cdot X'$$

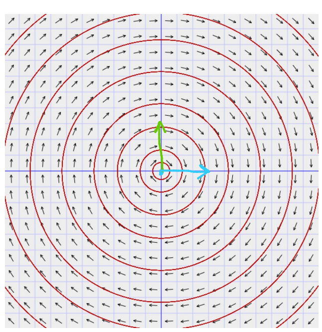
$$T^{-1}A = J\,T^{-1}$$

$$\underbrace{T^{-1} \cdot X'}_{} = T^{-1}AX$$

$$y' = J\,\underbrace{T^{-1}X}_{} = J y$$ → знамо да су кругови

$$T = \begin{bmatrix} \operatorname{Re}\gamma\downarrow & \operatorname{Im}\gamma\downarrow \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -2 & -\sqrt{6} \end{bmatrix}$$

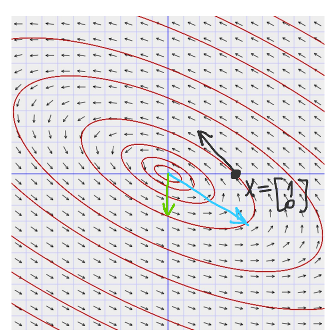
$J \curvearrowright$



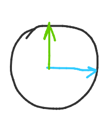
$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \overset{T}{\rightsquigarrow} T \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$

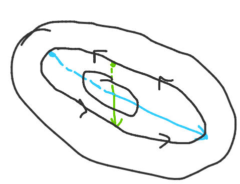
$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \overset{T}{\rightsquigarrow} T \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\sqrt{6} \end{bmatrix}$

$A \curvearrowright$



$X = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightsquigarrow X' = AX = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$





д) $A = \begin{bmatrix} -1 & -1 \end{bmatrix}$

д) $A = \begin{bmatrix} -1 & -1 \\ 2 & -3 \end{bmatrix}$

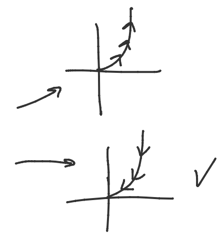
$\lambda_{1/2} = -2 \pm i$

$X^* = (0,0)$

$\alpha \pm i\beta \; (\alpha, \beta \neq 0)$ – спирала (фокус)

$\alpha > 0$: нестабилна спирала →

$\alpha < 0$: стабилна спирала → ✓



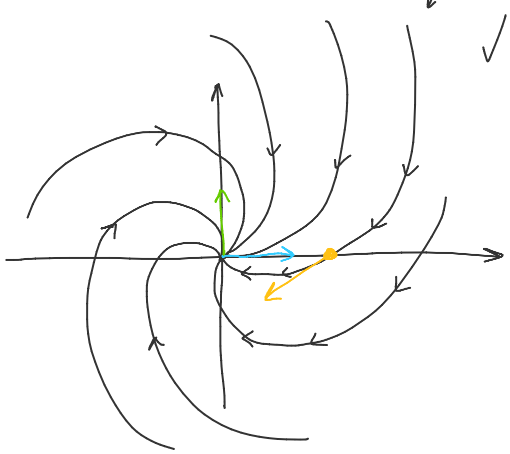
$A = T \cdot J \cdot T^{-1}$

$J = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix}$

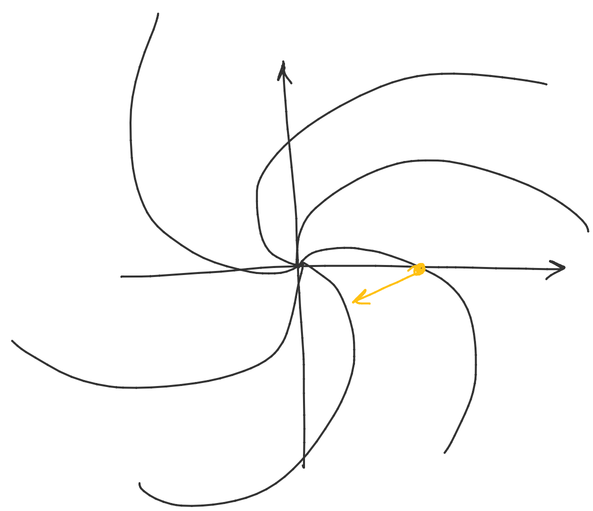
$(A - (-2+i)E)v = 0 \Rightarrow v = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$

$T = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$

J ↷

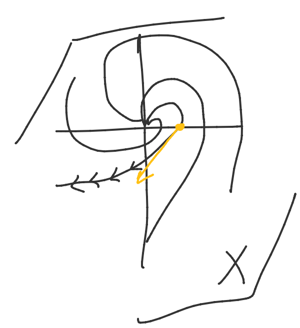


✓



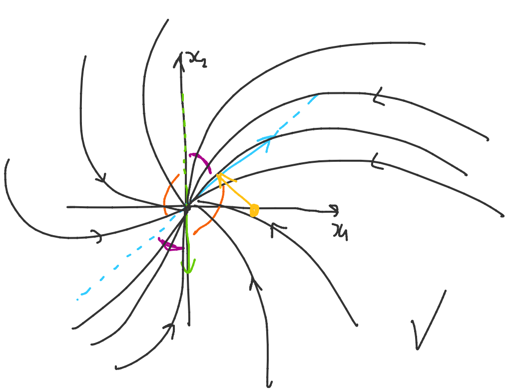
$X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$X' = JX = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$

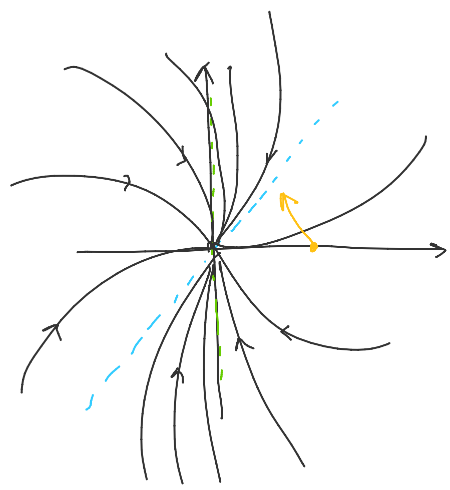


ређе

гушће

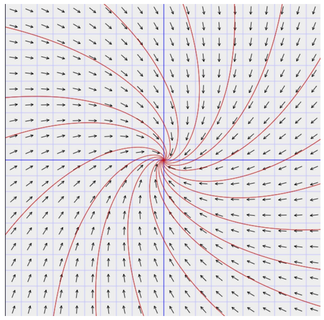


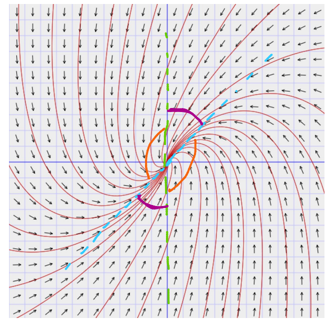
✓



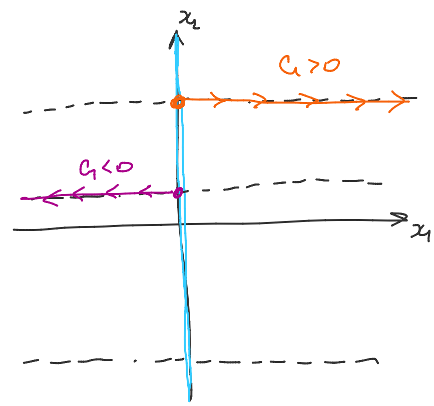
$X = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow X' = AX = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

J ↷ $\qquad$ A ↷





ђ) $A=\begin{bmatrix}5&0\\0&0\end{bmatrix}$



$X^*=?$

$\left.\begin{matrix}5x_1=0\\0=0\end{matrix}\right\} x_1=0$

$X^*\in\{(0,s)\mid s\in\mathbb{R}\}$ → неизоловани еквилибријуми (чворови)

$\left.\begin{matrix}x_1'=5x_1\\x_2'=0\end{matrix}\right\}$ $\begin{matrix}x_1=c_1e^{5t}\\x_2=c_2\end{matrix}$

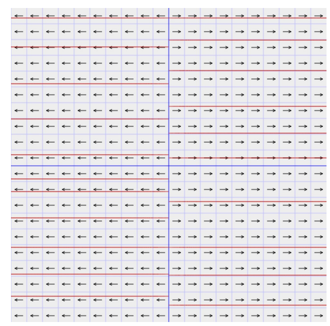
$c_2\in\mathbb{R}$

$c_1>0$: $t\to-\infty \Rightarrow x_1\to0$

$t\to\infty \Rightarrow x_1\to\infty$

$c_1<0$: $t\to-\infty \Rightarrow x_1\to0$

$t\to\infty \Rightarrow x_1\to-\infty$



ФТ су полуправе које почињу у неком X^*

1. Скицирати фазни портрет динамичког система $X'=AX$, ако је:

а) $A=\begin{bmatrix}2&0\\0&2\end{bmatrix}$ б) $A=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$ в) $A=\begin{bmatrix}3&1\\-4&-1\end{bmatrix}$ г) $A=\begin{bmatrix}-3&1\\-1&-1\end{bmatrix}$ д) $A=\begin{bmatrix}0&2\\0&0\end{bmatrix}$. → домаћи

б) $A=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}=(-1)\cdot E$

$\lambda_1=\lambda_2=-1$

$J=\begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix} \vee J=\begin{bmatrix}\lambda&1\\0&\lambda\end{bmatrix}$

✓

↓

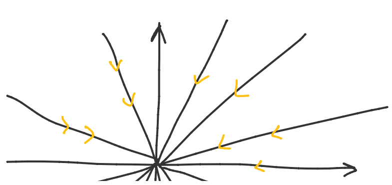
звезда

$X^*=(0,0)$

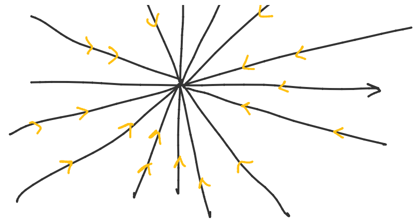
↳ стабилна звезда (сингуларни чвор)

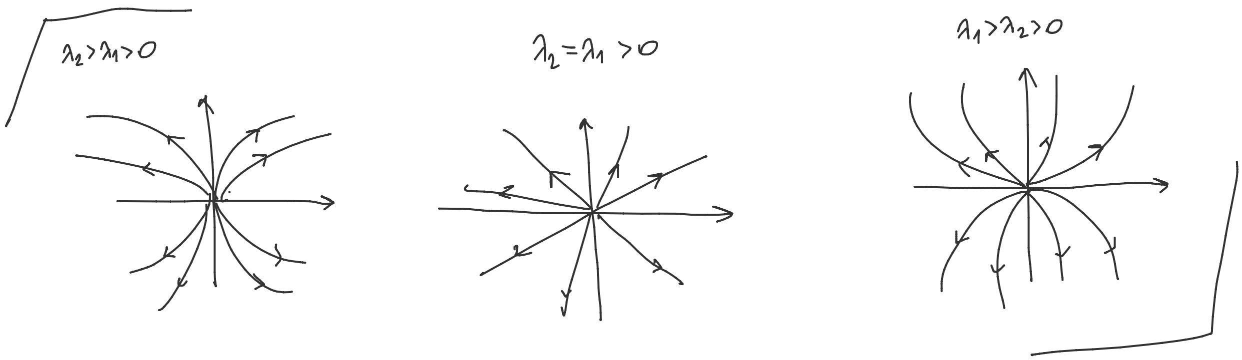
$\lambda<0$ – стабилно

$\lambda>0$ – нестабилно



$\lambda>0$ – нестабилно





г) $A=\begin{bmatrix}-3&1\\-1&-1\end{bmatrix}$ $\qquad \lambda_1=\lambda_2=-2 \rightsquigarrow$ сопс. век. $\gamma_1=\begin{bmatrix}1\\1\end{bmatrix} \Rightarrow$ 1 Ж. блок

$\lambda>0$: нестабилан

$J=\begin{bmatrix}-2&1\\0&-2\end{bmatrix} \to$ $(\lambda<0)$ стабилан дегенерисан чвор

$\hookrightarrow x_1'=-2x_1+x_2$

$x_2'=-2x_2$

$x_2\equiv 0$

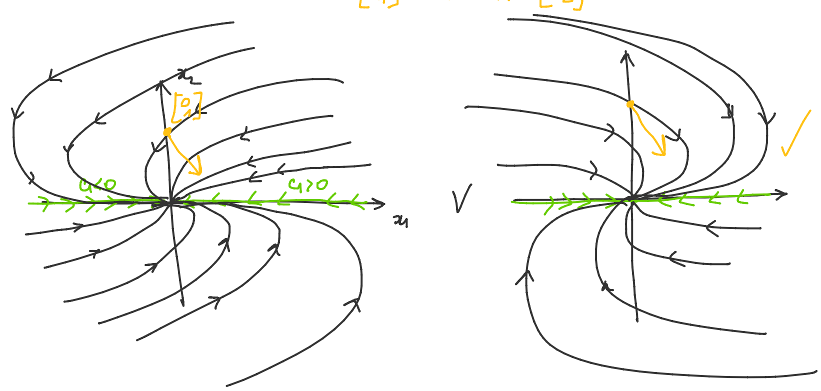
$x_1=c_1e^{-2t}$

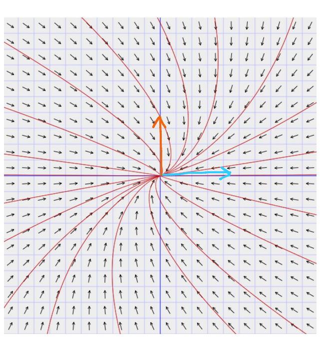
$(A+2E)\gamma_2=\gamma_1$

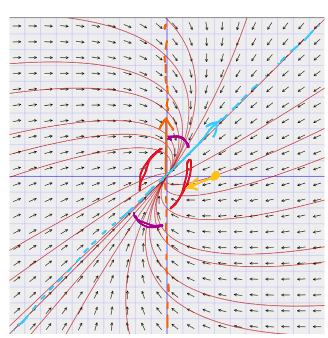
$\gamma_2=\begin{bmatrix}0\\1\end{bmatrix} \to$ уопштени с. век.

$T=\begin{bmatrix}1&0\\1&1\end{bmatrix}$

$X=\begin{bmatrix}0\\1\end{bmatrix} \rightsquigarrow X'=JX=\begin{bmatrix}1\\-2\end{bmatrix}$







густе

ређе

$X=\begin{bmatrix}1\\0\end{bmatrix}, X'=AX=\begin{bmatrix}-3\\-1\end{bmatrix}$

2. Нека су $a, b \in \mathbb{R}$ и нека је $a \neq \pm b$. Свести диференцијалну једначину $y'' + 2ay' + b^2y = 0$ на систем диференцијалних једначина. У зависности од параметара a и b испитати тип еквилибријума и скицирати фазне портрете.

$$y'' = -2ay' - b^2 y$$

$$\left.\begin{array}{l} x_1 = y \\ x_2 = y' \end{array}\right\} \quad \begin{array}{l} x_1' = y' = x_2 \\ x_2' = -2ay' - b^2 y = -2ax_2 - b^2 x_1 \end{array} \quad , X' = AX, \ A = \begin{bmatrix} 0 & 1 \\ -b^2 & -2a \end{bmatrix}$$

$$\det(A - \lambda E) = 0$$

$$(0-\lambda)(-2a-\lambda) - (-b^2)\cdot 1 = 0$$

$$\lambda^2 + 2a\lambda + b^2 = 0$$

$$D = (2a)^2 - 4b^2 = 4(a^2 - b^2) \neq 0 \quad (a \neq \pm b)$$

1° $a^2 - b^2 > 0$, $D > 0$, $\lambda_{1/2} = \dfrac{-2a \pm \sqrt{D}}{2} = -a \pm \sqrt{a^2 - b^2}$

$(\lambda_1 \neq \lambda_2)$

2° $b^2 - a^2 > 0$, $D < 0$, $\lambda_{1/2} = \dfrac{-2a \pm i\sqrt{-D}}{2} = -a \pm i\sqrt{b^2 - a^2}$

$$X^* = ? \quad \left.\begin{array}{l} x_2 = 0 \\ -b^2 x_1 - 2ax_2 = 0 \end{array}\right\} \quad -b^2 x_1 = 0 \quad \begin{array}{l} \nearrow b = 0 \\ \searrow x_1 = 0 \end{array}$$

$b = 0$: $(s, 0)$, $s \in \mathbb{R}$

$b \neq 0$: $(0, 0)$

1° $a^2 - b^2 > 0$ ($a = 0$ немогуће)

1.1° $b = 0$ $\quad X^* \in \{(s, 0) \mid s \in \mathbb{R}\}$ — неизоловани еквилибријуми

у свим осталим случ. је $X^* = (0, 0)$

1.2° $b \neq 0$

$$\lambda_1 = -a - \sqrt{a^2 - b^2} < -a + \sqrt{a^2 - b^2} = \lambda_2$$

$1.2.1^\circ$ $a > 0$

$\lambda_1 < 0$

$\lambda_2 < 0 \iff -a + \sqrt{a^2 - b^2} < 0$

$\sqrt{a^2 - b^2} < a \quad /^2 \quad$ (обе > 0)

$a^2 - b^2 < a^2$

$-b^2 < 0 \quad \checkmark$

$\lambda_1 < \lambda_2 < 0 \to$ стабилан чвор

$1.2.2^\circ$ $a < 0$

$\lambda_2 > 0$

1.2.2° $a<0$

$\lambda_2>0$

$\lambda_1>0 \Leftrightarrow -a-\sqrt{a^2-b^2}>0 \Leftrightarrow -a>\sqrt{a^2-b^2} \,/^2$ (обе >0)

$a^2>a^2-b^2$ ✓

$\lambda_2>\lambda_1>0 \rightarrow$ нестабилан чвор

2° $b^2-a^2>0$ ($b=0$ немогуће)

$\lambda_{1/2}=-a\pm i\sqrt{b^2-a^2}$

2.1° $a=0$

$\lambda_{1/2}=\pm i\sqrt{b^2-a^2} \rightarrow$ центар

2.2° $a>0$ $\quad Re(\lambda_1)=Re(\lambda_2)<0 \rightarrow$ стабилан фокус

2.3° $a<0$ $\quad Re(\lambda_1)=Re(\lambda_2)>0 \rightarrow$ нестабилни фокус