(1) $A \in M_2(\mathbb{R})$

$$B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad y' = B^{-1}ABy + D(x)$$

$$e^{xA} = \begin{bmatrix} e^x & 0 \\ -3e^x+3e^{2x} & e^{2x} \end{bmatrix}$$

а) $D(x) \equiv 0$, решити систем

б) Одредити $B^{-1}AB$.

в) $D(x) = \begin{bmatrix} 1 \\ e^x \end{bmatrix}$, решити систем.

а) $y' = \underbrace{B^{-1}AB}_{C}y$

$$\text{ОР}: \ y(x) = e^{xC} \cdot k = e^{x(B^{-1}AB)} \cdot k = B^{-1}e^{xA}B \cdot k = \underbrace{\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^x & 0 \\ -3e^x+3e^{2x} & e^{2x} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}_{e^{xC} = \begin{bmatrix} 7e^x-6e^{2x} & 14e^x-14e^{2x} \\ -3e^x+3e^{2x} & -6e^x+7e^{2x} \end{bmatrix}} \cdot k, \ k \in \mathbb{R}^2$$

$$B^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

б) $e^{xC} \rightsquigarrow C$?

$\Phi(x)$ – фундаментална матрица, важи:
$y' = Ay$

$$\Phi(x) = [\varphi_1\downarrow \ \dots \ \varphi_n\downarrow]$$

$$\Phi'(x) = [\varphi_1' \ \dots \ \varphi_n'] = [A\varphi_1 \ \dots \ A\varphi_n] = A[\varphi_1 \ \dots \ \varphi_n] = A \cdot \Phi(x)$$

$$W(x) \neq 0$$

за исдјкк e^{xA} је фунд. матр:

1) $\Phi'(x) = (e^{xA})' = Ae^{xA} = A\Phi(x)$

2) $W(x) = \det(e^{xA}) = e^{tr(xA)} \neq 0$

$$e^{xC} = \Phi(x)$$

$$\Phi'(x) = C \cdot \Phi(x) \Rightarrow C = \Phi'(x) \cdot \overset{W(x)\neq 0}{\Phi^{-1}(x)}$$

$$\Phi'(x) = C\cdot\Phi(x) \overset{W(x)\neq 0}{\Rightarrow} C = \Phi'(x)\cdot\Phi^{-1}(x)$$

$$\Phi'(x) = \begin{bmatrix} 7e^{x}-12e^{2x} & 14e^{x}-28e^{2x} \\ -3e^{x}+6e^{2x} & -6e^{x}+14e^{2x} \end{bmatrix}$$

$$\Phi^{-1}(x) = \left(e^{xC}\right)^{-1} \underset{\text{за вежбу}}{=} e^{-xC} = \Phi(-x) = \begin{bmatrix} 7e^{-x}-6e^{-2x} & 14e^{-x}-14e^{-2x} \\ -3e^{-x}+3e^{-2x} & -6e^{-x}+7e^{-2x} \end{bmatrix}$$

$$C = B^{-1}AB = \Phi'(x)\cdot\Phi^{-1}(x) = \dots = \begin{bmatrix} -5 & -14 \\ 3 & 8 \end{bmatrix}.$$

B) $y(x) = \Phi(x)\cdot\left(C + \int \Phi^{-1}(x)\cdot D(x)\,dx\right)$

$\Phi^{-1}(x) = \Phi(-x)$

за e^{xC}

$$y(x) = \Phi(x)\cdot C + \begin{bmatrix} -4 + e^{x}(14x+14) \\ \frac{3}{2} + e^{x}(-6x-7) \end{bmatrix}.$$

2.

$x^2 yy' = \frac{y^2}{2} + e^{2z} \quad /\cdot 2 \quad \Rightarrow \quad x^2\cdot(2yy') = y^2 + 2e^{2z}$

$2x^2 z' = -3\frac{y^2}{e^{2z}} - 4 \quad /\cdot e^{2z} \quad \Rightarrow \quad x^2\cdot(2e^{2z}\cdot z') = -3y^2 - 4e^{2z}$

$y_1 = y^2 \rightsquigarrow y_1' = 2yy'$

$z_1 = e^{2z} \longrightarrow z_1' = e^{2z}\cdot 2z'$

$$x^2\cdot y_1' = y_1 + 2z_1$$
$$x^2\cdot z_1' = -3y_1 - 4z_1$$

$y_1(x) \rightsquigarrow y_1(t)$

$z_1(x) \quad\;\; z_1(t)$

$z_{1t}' = \frac{dz_1}{dt} = x^2\cdot z_1' = x^2\cdot\frac{dz_1}{dx} = x^2\cdot\frac{dz_1}{dt}\cdot\frac{dt}{dx}$

$x^2\cdot\frac{dt}{dx} = 1 \Rightarrow \frac{dt}{dx} = \frac{1}{x^2} \Rightarrow t = -\frac{1}{x}.$

$$y_{1t}' = y_1 + 2z_1$$
$$z_{1t}' = -3y_1 - 4z_1$$

(ЛСДЈКК)

...

$$\begin{bmatrix} y_1(t) \\ z_1(t)\end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3\end{bmatrix}\begin{bmatrix} e^{-t} & \\ & e^{-2t}\end{bmatrix}\cdot\begin{bmatrix} c_1\\ c_2\end{bmatrix}, \quad c_1,c_2\in\mathbb{R}$$

$$y_1(t) = c_1e^{-t} - 2c_2e^{-2t} \rightsquigarrow y_1(x) = c_1e^{1/x} - 2c_2 e^{2/x}$$

$$y(x) = \pm\sqrt{c_1e^{1/x} - 2c_2e^{2/x}}$$

Фазни портрет

$y(x) \rightsquigarrow X(t)$

$X' = AX$ – динамички систем

t – време

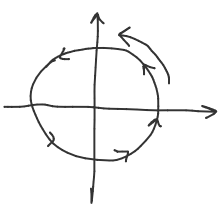
$A\in M_2(\mathbb{R})$

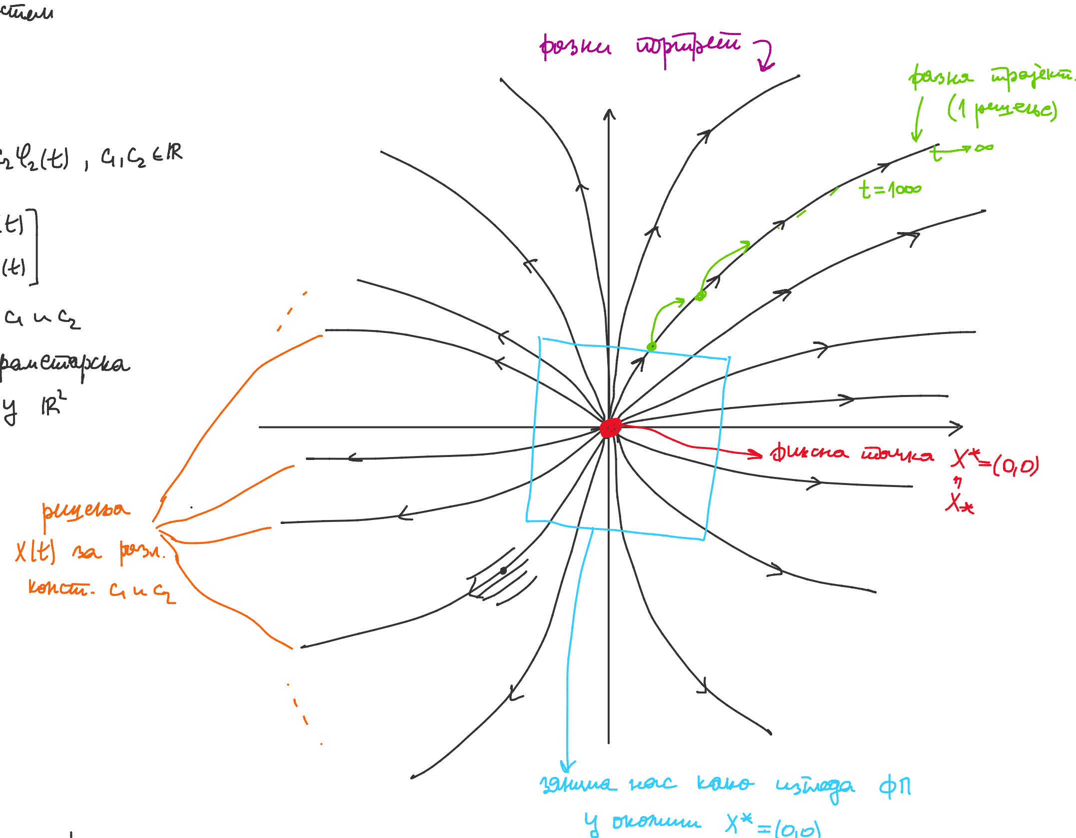
решења: $X(t) = c_1\varphi_1(t)+c_2\varphi_2(t)$, $c_1,c_2\in\mathbb{R}$

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t)\end{bmatrix}$$

↳ за фикс. c_1 и c_2 ово је параметарска крива у $\mathbb{R}^2$

нпр. $\begin{bmatrix} \cos t \\ \sin t\end{bmatrix}$:





фиксна тачка = еквилибријум $\quad X'=0$

$$X' = F(X) \rightsquigarrow F(X^*) = 0$$

ЛСДЈКК: $\quad A\cdot X^* = 0 \Rightarrow$ реш. је вект. потпр. од $\mathbb{R}^2$:
- тачка $(0,0)$ $\quad (\dim = 0)$
- права кроз $(0,0)$ $\quad (\dim = 1)$
- цео $\mathbb{R}^2$ $\quad (\dim = 2)$

1. Скицирати фазни портрет динамичког система $X' = AX$, ако је:

а) $A = \begin{bmatrix} -1 & 2 \\ 0 & -3\end{bmatrix}$; б) $A = \begin{bmatrix} -1 & 3 \\ 5 & -3\end{bmatrix}$; в) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}$;

г) $A = \begin{bmatrix} -2 & -5 \\ 2 & 2\end{bmatrix}$; д) $A = \begin{bmatrix} -1 & -1 \\ 2 & -3\end{bmatrix}$; ђ) $A = \begin{bmatrix} 5 & 0 \\ 0 & 0\end{bmatrix}$.

a) $\lambda_1=-1,\ \lambda_2=-3$

$$X'=0 \Rightarrow \left.\begin{aligned}-x_1+2x_2&=0\\ -3x_2&=0\end{aligned}\right\} \Rightarrow x_1=x_2=0 \Rightarrow X^*=(0,0).$$

$x_1'=x_2'=0$

$\underline{\lambda_1=-1}$ $\rightsquigarrow (A+E)\gamma_1=0$

$\hookrightarrow \gamma_1=\begin{bmatrix}1\\0\end{bmatrix}$

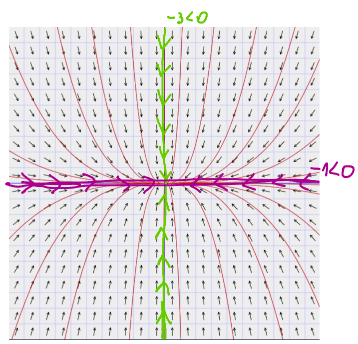
$\underline{\lambda_2=-3}$ $(A+3E)\gamma_2=0$

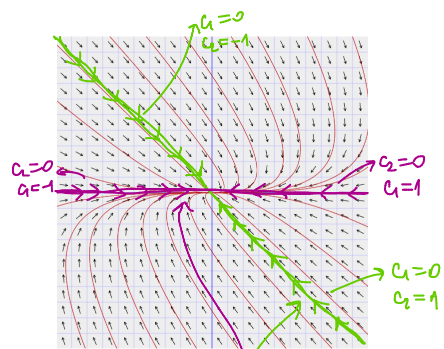
$\hookrightarrow \gamma_2=\begin{bmatrix}1\\-1\end{bmatrix}$

$$X(t)=c_1e^{-t}\begin{bmatrix}1\\0\end{bmatrix}+c_2e^{-3t}\begin{bmatrix}1\\-1\end{bmatrix}$$

λ_1,λ_2 – негативни и различни $\longrightarrow$ стабилан чвор

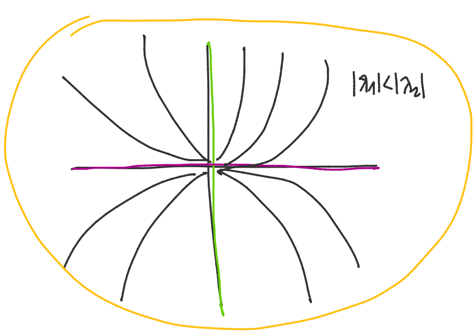
$$J=\begin{bmatrix}\underline{-1} & 0\\ 0 & \underline{-3}\end{bmatrix} \qquad A=\begin{bmatrix}-1 & 2\\ 0 & -3\end{bmatrix}$$





$c_1=0$: $c_2=1$ $(c_2=-1)$ $\quad X(t)=\begin{bmatrix}e^{-3t}\\-e^{-3t}\end{bmatrix},\ x_2=-x_1<0$

$c_2=0$, $c_1=1$ $(c_1=-1)$ $\quad X(t)=\begin{bmatrix}\pm e^{-t}\\0\end{bmatrix},\ x_2=0$



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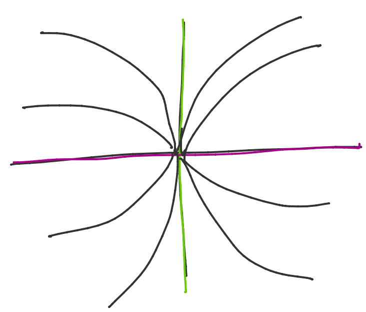
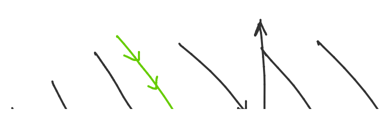
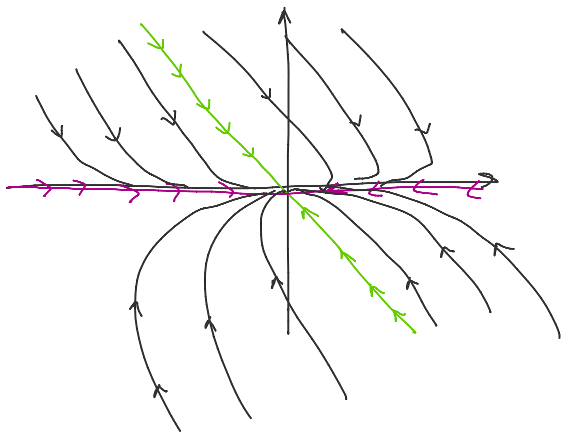


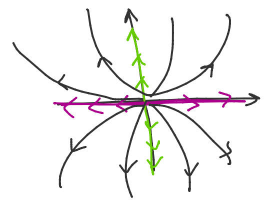


$\Gamma\ \lambda_2\neq\lambda_1,\ \lambda_1,\lambda_2>0$



$\lambda_2 \neq \lambda_1$, $\lambda_1, \lambda_2 > 0$

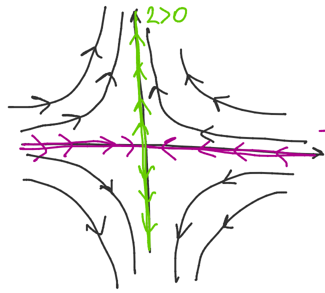
нестабилан чвор



б) $A=\begin{bmatrix}-1 & 3\\ 5 & -3\end{bmatrix}$, $X^* = (0,0)$

$\lambda_1 = -6$
$\lambda_2 = 2$

седло



$\dot{J} = \begin{bmatrix}-6 & \\ & 2\end{bmatrix}$

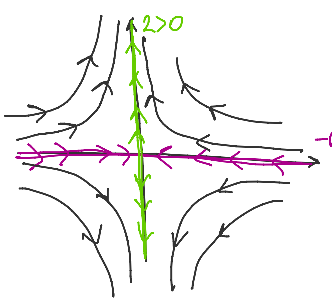
$\lambda_1 \rightsquigarrow v_1 = \begin{bmatrix}3\\-5\end{bmatrix}$

$\lambda_2 \rightsquigarrow v_2 = \begin{bmatrix}1\\1\end{bmatrix}$

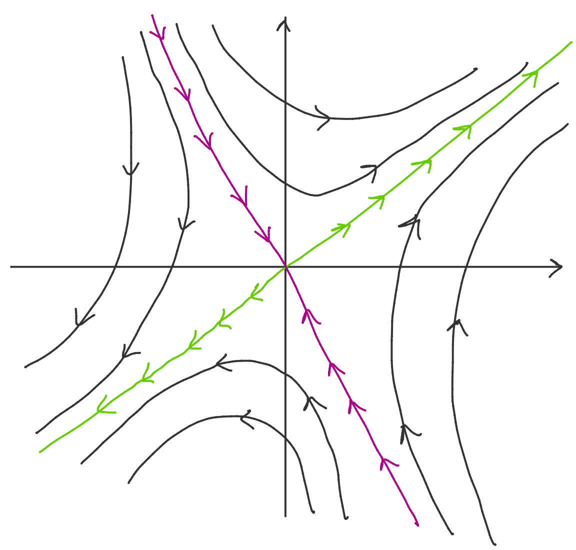
$$X(t) = c_1 e^{-6t}\begin{bmatrix}3\\-5\end{bmatrix} + c_2 e^{2t}\begin{bmatrix}1\\1\end{bmatrix}$$

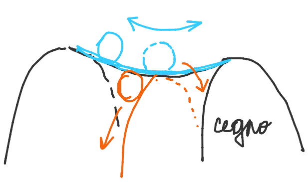
$c_1 = 0$, $c_2 = 1$ $\quad X(t) = \begin{bmatrix}e^{2t}\\e^{2t}\end{bmatrix}$, $x_1 = x_2$

$c_2 = 0$, $c_1 = 1$ $\quad X(t) = \begin{bmatrix}3e^{-6t}\\-5e^{-6t}\end{bmatrix}$, $5x_1 = -3x_2$

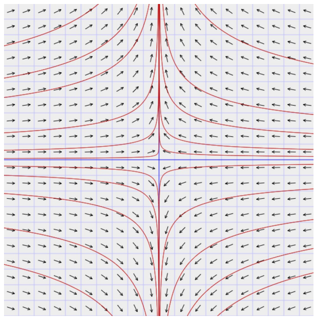


$\rightsquigarrow$

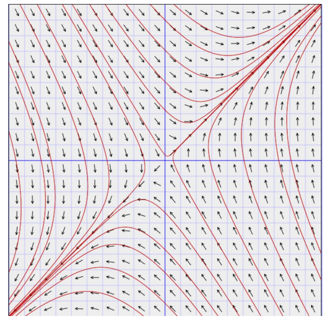




$\dot{J}$ ↷



A ↷



В) $A = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$, $X^* = (0,0)$

↳ $\lambda_{1/2} = \pm i = \alpha \pm i\beta$

$\underline{\alpha = 0}$ – центар

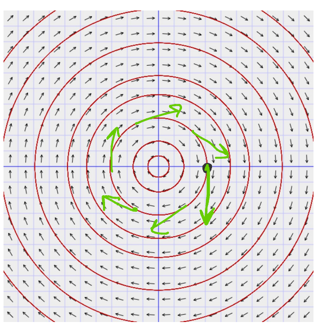
$e^{xA} = R_x$

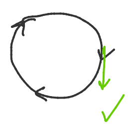
$x_1(t)$

$$X(t) = c_1 \begin{bmatrix}\cos t\\ -\sin t\end{bmatrix} + c_2 \begin{bmatrix}\sin t\\ \cos t\end{bmatrix}$$

$x_2(t)$

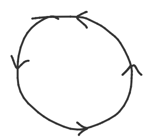
$x_1^2 + x_2^2 = \text{const} \to$ кругови $\to$ периодичне





✓

V



?

(1,0) – шта је оријентација извода?

$X' = AX$

↳ $\left.\begin{matrix}x_1 = 1\\ x_2 = 0\end{matrix}\right\}$ $X' = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}0\\ -1\end{bmatrix}$

↳ $X = \begin{bmatrix}1\\ 0\end{bmatrix}$