Нехомогени ЛСДЈКК

$$y'(x) = \underbrace{AY(x)}_{\text{хомогени део}} + \overbrace{g(x)}^{\text{нехомогени део}}$$

$$\text{ОР}: Y(x) = \underbrace{Y_H(x)}_{\text{опште реш. хом.}} + \underbrace{Y_P(x)}_{\text{партикуларно нехомогене}}$$

$$(*)\quad g(x) = P_s[x]\cdot e^{\mu x}, \quad \mu \in \mathbb{R}$$

$$s = st(P_s[x]) = \deg(P_s[x]) \in \mathbb{N}_0$$

$$\Rightarrow Y_P(x) = Q_{m+s}[x]\cdot e^{\mu x}$$

m = вишеструкост броја μ као решења карактеристичне једначине од A

(1) а)
$$y_1' = 2y_1 + y_2 + xe^x$$
$$y_2' = -y_1 + 2y_2 - e^x$$

б)
$$y_1' = 2y_1 + y_2 + 2e^x$$
$$y_2' = y_1 + 2y_2 - 3e^{4x}$$

а)
$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$Y' = AY + g$$

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

$$g = \begin{bmatrix} xe^x \\ -e^x \end{bmatrix} = \begin{bmatrix} x \\ -1 \end{bmatrix}\cdot e^x$$

ХОМОГЕНА: $Y' = AY$

$$\det(A-\lambda E) = 0 \Rightarrow \lambda_{1/2} = 2 \pm i$$

$$\lambda_1 = 2+i \rightarrow (A-\lambda_1 E)\gamma_1 = 0 \Rightarrow \gamma_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\varphi(x) = \gamma_1 \cdot e^{\lambda_1 x} = e^{2x}\cdot \begin{bmatrix} \cos x + i\sin x \\ i\cos x - \sin x \end{bmatrix} \quad \begin{matrix} \nearrow \mathrm{Re} \\ \searrow \mathrm{Im} \end{matrix}$$

$$\text{ОРХ}: Y_H(x) = c_1 e^{2x}\begin{bmatrix} \cos x \\ -\sin x \end{bmatrix} + c_2 e^{2x}\begin{bmatrix} \sin x \\ \cos x \end{bmatrix}.$$

$c_1, c_2 \in \mathbb{R}$

НЕХОМОГЕНИ: $g(x) = \begin{bmatrix} x \\ -1 \end{bmatrix}\cdot \underset{e^{\mu x}}{\underset{\parallel}{e^x}}$. ($\mu = 1$ није реш. $\det(A-\lambda E) = 0$)

НЕХОМОГЕНИ: $g(x) = \begin{bmatrix} x \\ -1 \end{bmatrix} \cdot e^{x}$ $\quad$ $e^{\mu x}$

$\overset{\shortparallel}{P_s}[x]$ $\quad$ $\mu = 1$

$s = 1$

$\mu = 1$ није реш. $\det(A - \lambda E) = 0$ $(\lambda_{1/2} = 2 \pm i)$ $\Big\}\, m = 0$

(μ није сопс. вр. од A)

$$y_p(x) = Q_{s+m}[x] \cdot e^{\mu x} = Q_1[x] \cdot e^{x} = \begin{bmatrix} a_1 x + b_1 \\ a_2 x + b_2 \end{bmatrix} \cdot e^{x} \Rightarrow y_p' = Ay_p + g$$

$$\begin{bmatrix} a_1 x + b_1 + a_1 \\ a_2 x + b_2 + a_2 \end{bmatrix} \cdot \cancel{e^{x}} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} a_1 x + b_1 \\ a_2 x + b_2 \end{bmatrix} \cdot \cancel{e^{x}} + \begin{bmatrix} x \\ -1 \end{bmatrix} \cdot \cancel{e^{x}}$$

$$\begin{bmatrix} a_1 x + b_1 + a_1 \\ a_2 x + b_2 + a_2 \end{bmatrix} = \begin{bmatrix} 2a_1 x + 2b_1 + a_2 x + b_2 + x \\ -a_1 x - b_1 + 2a_2 x + 2b_2 - 1 \end{bmatrix} \quad \overset{\forall x \in \mathbb{R}}{\Longrightarrow}$$

$a_1 = 2a_1 + a_2 + 1$

$b_1 + a_1 = 2b_1 + b_2$

$a_2 = -a_1 + 2a_2$

$b_2 + a_2 = -b_1 + 2b_2 - 1$

$a_2 - a_1 = 0$

$a_1 + a_2 = -1$

$a_2 = -\frac{1}{2},\ a_1 = -\frac{1}{2}$

$b_1 + b_2 = -\frac{1}{2}$

$b_2 - b_1 = \frac{1}{2}$

$b_2 = 0,\ b_1 = -\frac{1}{2}$

$$y_p(x) = \begin{bmatrix} -\frac{x}{2} - \frac{1}{2} \\ -\frac{x}{2} \end{bmatrix} \cdot e^{x}$$

ОР: $y(x) = y_H(x) + y_p(x)$.

б) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ $\quad\quad$ $y' = Ay + g$

$g = \begin{bmatrix} 2e^{x} \\ -3e^{4x} \end{bmatrix} \rightsquigarrow g(x) = P_s[x] \cdot e^{\mu x}$ није могуће!

$\underline{g(x)} = \underline{g_1(x)} + \underline{g_2(x)} =$

$= \begin{bmatrix} 2e^{x} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -3e^{4x} \end{bmatrix}$

$A \rightsquigarrow \lambda_1 = 1,\ \lambda_2 = 3$

$\gamma_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad$ $\gamma_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$g_1 = \begin{bmatrix} 2e^{x} \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \cdot e^{x} \longrightarrow y_{p_1}(x)$

$\gamma_1 = \begin{bmatrix} \\ -1 \end{bmatrix} \quad \gamma_2 = \begin{bmatrix} \\ 1 \end{bmatrix}$

$\varphi_1(x) = \begin{bmatrix} e^{x} \\ -e^{x} \end{bmatrix}, \quad \varphi_2(x) = \begin{bmatrix} e^{3x} \\ e^{3x} \end{bmatrix}$

$g_1 = \begin{bmatrix} 2e^{x} \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \cdot e^{x} \rightsquigarrow y_{P_1}(x)$

$g_2 = \begin{bmatrix} 0 \\ -3e^{4x} \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix} \cdot e^{4x} \rightsquigarrow y_{P_2}(x)$

$y_P(x) = y_{P_1}(x) + y_{P_2}(x)$

$\underline{g_1(x):}$

$g_1(x) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \cdot e^{x} \Rightarrow s = 0$

$\mu = 1 \xrightarrow[(\lambda_1 = 1)]{} m = 1$

$\Rightarrow y_{P_1}(x) = Q_1[x] \cdot e^{x} = \begin{bmatrix} a_1 x + b_1 \\ a_2 x + b_2 \end{bmatrix} \cdot e^{x}$

$y_{P_1}' = A y_{P_1} + g_1$

$$\cancel{e^{x}} \cdot \begin{bmatrix} a_1 x + b_1 + a_1 \\ a_2 x + b_2 + a_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a_1 x + b_1 \\ a_2 x + b_2 \end{bmatrix} \cdot \cancel{e^{x}} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \cdot \cancel{e^{x}}$$

$a_1 = 2a_1 + a_2$

$b_1 + a_1 = 2b_1 + b_2 + 2$

$a_2 = a_1 + 2a_2$

$b_2 + a_2 = b_1 + 2b_2$

$a_1 + a_2 = 0$

$a_1 + a_2 = 0$

$a_2 = -a_1$

$b_1 + b_2 = a_1 - 2$

$b_1 + b_2 = a_2 = -a_1$

$a_1 - 2 = -a_1 \Rightarrow a_1 = 1 \Rightarrow b_1 + b_2 = -1$

$b_2 = -1 - b_1$

np. $b_1 = 0$

$b_2 = -1$

$y_{P_1}(x) = \begin{bmatrix} x \\ -x - 1 \end{bmatrix} \cdot e^{x}$

$\underline{g_2(x):}$

$g_2(x) = \begin{bmatrix} 0 \\ -3 \end{bmatrix} \cdot e^{4x} \Rightarrow s = 0$

$\mu = 4 \Rightarrow m = 0$

$\Rightarrow y_{P_2} = Q_0[x] \cdot e^{4x} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \cdot e^{4x}$

$y_{P_2}' = A y_{P_2} + g_2$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \cdot 4 \cdot \cancel{e^{4x}} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \cdot \cancel{e^{4x}} + \begin{bmatrix} 0 \\ -3 \end{bmatrix} \cdot \cancel{e^{4x}}$$

$4\alpha = 2\alpha + \beta$

$4\beta = \alpha + 2\beta - 3$

$$\beta - 2\alpha = 0 \qquad \Rightarrow \beta = 2\alpha$$
$$2\beta - \alpha + 3 = 0 \qquad 4\alpha - \alpha + 3 = 0$$
$$\alpha = -1,\ \beta = -2$$

$$y_{p_2} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \cdot e^{4x}$$

$$\text{OP}: \ y(x) = \underbrace{c_1 \cdot \begin{bmatrix} e^{x} \\ -e^{x} \end{bmatrix} + c_2 \cdot \begin{bmatrix} e^{3x} \\ e^{3x} \end{bmatrix}}_{y_H(x)} + \underbrace{\begin{bmatrix} x \\ -x-1 \end{bmatrix} \cdot e^{x} + \begin{bmatrix} -1 \\ -2 \end{bmatrix} \cdot e^{4x}}_{y_P(x)}, \quad c_1, c_2 \in \mathbb{R}$$

② $$y_1' = y_2 + tg^2 x + 1 \quad ?$$
$$y_2' = -y_1 + tg\, x$$

$$y' = A(x) \cdot y$$
$$\text{OP}: \ y(x) = \underbrace{\Phi(x)}_{\text{фундаментална матрица}} \cdot c, \ c \in \mathbb{R}^n$$

$$y' = A(x) \cdot y + g(x)$$
$$y(x) = \Phi(x) \cdot \left(c + \int \Phi^{-1}(x) \cdot g(x)\, dx\right)$$
$$= \underbrace{\boxed{\Phi(x) \cdot c}}_{y_H(x)} + \underbrace{\boxed{\Phi(x) \cdot \int \Phi^{-1}(x) g(x)\, dx}}_{y_P(x)}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$g(x) = \begin{bmatrix} tg^2 x + 1 \\ tg\, x \end{bmatrix} = \begin{bmatrix} \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} \\ tg\, x \end{bmatrix} = \begin{bmatrix} \frac{1}{\cos^2 x} \\ tg\, x \end{bmatrix}.$$

$$y_H? \longleftarrow \Phi(x)$$

$$y_1' = y_2$$
$$y_2' = -y_1$$

$$y_1'' = y_2' = -y_1 \Rightarrow y_1'' + y_1 = 0 \rightsquigarrow \left.\begin{aligned} y_1 &= c_1 \cos x + c_2 \sin x \\ y_2 = y_1' &= -c_1 \sin x + c_2 \cos x \end{aligned}\right\} \Rightarrow y = c_1 \cdot \underbrace{\begin{bmatrix} \cos x \\ -\sin x \end{bmatrix}}_{\varphi_1(x)} + c_2 \cdot \underbrace{\begin{bmatrix} \sin x \\ \cos x \end{bmatrix}}_{\varphi_2(x)}$$

$$\Phi(x) = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightsquigarrow A^{-1} = \frac{1}{ad - bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$W(x) = \det \Phi(x) = \cos^2 x - (-\sin^2 x) = 1$$

$$\Phi^{-1}(x) = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$$

$$\int \Phi^{-1}(x) \cdot g(x)\, dx = \int \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\cos^2 x} \\ tg\, x \end{bmatrix} dx =$$

$$\int \begin{bmatrix} \frac{1}{\ldots} - \frac{\sin^2 x}{\ldots} \end{bmatrix} \qquad \int \begin{bmatrix} \cos x \end{bmatrix}$$

$$= \int \begin{bmatrix} \frac{1}{\cos x} - \frac{\sin^2 x}{\cos x} \\ \frac{\sin x}{\cos^2 x} + \sin x \end{bmatrix} dx = \int \begin{bmatrix} \cos x \\ \sin x + \frac{\sin x}{\cos^2 x} \end{bmatrix} dx =$$

$$= \begin{bmatrix} \int \cos x \, dx \\ \int \left(\sin x + \frac{\sin x}{\cos^2 x}\right) dx \end{bmatrix} = \begin{bmatrix} \sin x \\ -\cos x + \frac{1}{\cos x} \end{bmatrix} = \begin{bmatrix} \sin x \\ \frac{1-\cos^2 x}{\cos x} \end{bmatrix} = \begin{bmatrix} \sin x \\ \frac{\sin^2 x}{\cos x} \end{bmatrix}$$

$$\int \frac{\sin x}{\cos^2 x} dx = \int \frac{-dt}{t^2} = \frac{1}{t} + C = \frac{1}{\cos x} + C$$

$\cos x = t$

$dt = -\sin x \, dx$

$$y_p(x) = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \cdot \begin{bmatrix} \sin x \\ \frac{\sin^2 x}{\cos x} \end{bmatrix} = \begin{bmatrix} \cos x \sin x + \frac{\sin^3 x}{\cos x} \\ -\sin^2 x + \sin^2 x \end{bmatrix} = \begin{bmatrix} \frac{\sin x}{\cos x}(\cos^2 x + \sin^2 x) \\ 0 \end{bmatrix} = \begin{bmatrix} \operatorname{tg} x \\ 0 \end{bmatrix}.$$

OP: $y(x) = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \cdot C + \begin{bmatrix} \operatorname{tg} x \\ 0 \end{bmatrix}, \quad C \in \mathbb{R}^2.$

Експоненти матрице

$A \in M_n(\mathbb{R})$ $(\mathbb{C})$ $\rightsquigarrow$ $\exp : M_n(\mathbb{R}) \to M_n(\mathbb{R})$

$$\exp(A) = e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

$$\left[e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \right]$$

$y' = py \rightsquigarrow y(x) = e^{px} \cdot c$

$y' = Ay \rightsquigarrow y(x) = e^{Ax} \cdot c$

Особине експонента: $(\forall A, B \in M_n(\mathbb{R}))$

$\boxed{(1)}$ $\frac{d}{dx}(e^{xA}) = Ae^{xA}$

$\boxed{(2)}$ $AB = BA \Rightarrow e^{A+B} = e^A \cdot e^B$

$\boxed{(3)}$ $AB = BA \Rightarrow Ae^B = e^B A$

$\boxed{(4)}$ $\det(e^A) = e^{\operatorname{tr} A}$

$\boxed{(5)}$ $\lim\limits_{n\to\infty}\left(E+\frac{A}{n}\right)^n = e^A$

(3) Испитати да ли постоји реална матрица A таква да је:

а) $e^A = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$; б) $e^A = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}$

$e^x = 4 > 0 \Rightarrow x = \ln 4$

$e^x = -2 \leq 0$ ✗

а) $e^A \in M_2(\mathbb{R}) \Rightarrow A \in M_2(\mathbb{R}) \Rightarrow trA \in \mathbb{R}$

$$(4) \Rightarrow \left.\begin{aligned} \det(e^A) &= e^{trA} > 0 \\ &\parallel \\ \det\left(\begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}\right) &= -4 < 0 \end{aligned}\right\} \lightning \quad \nexists$$

$\det > 0$ потребан, али не и довољан!

$$б) \left.\begin{aligned} \det(e^A) &= e^{trA} \\ &\parallel \\ \det\left(\begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}\right) &= 4 \end{aligned}\right\} e^{trA} = 4, \quad \underline{\underline{trA = \ln 4}}.$$

$(3) \Rightarrow A = B$: $AB = AA = A^2 = AA = BA \Rightarrow Ae^A = e^A A$

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \qquad \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

$\cancel{-\alpha = -\alpha}$

$\left.\begin{aligned} -4\beta &= -\beta \\ -\gamma &= -4\gamma \end{aligned}\right\} \Rightarrow \beta = \gamma = 0$

$\cancel{-4\delta = -4\delta}$

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}, A^2 = \begin{bmatrix} \alpha^2 & 0 \\ 0 & \delta^2 \end{bmatrix}, \ldots, A^k = \begin{bmatrix} \alpha^k & 0 \\ 0 & \delta^k \end{bmatrix}$$

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \frac{\begin{bmatrix} \alpha^k & 0 \\ 0 & \delta^k \end{bmatrix}}{k!} = \begin{bmatrix} \sum\limits_{k=0}^{\infty} \frac{\alpha^k}{k!} & 0 \\ 0 & \sum\limits_{k=0}^{\infty} \frac{\delta^k}{k!} \end{bmatrix} = \begin{bmatrix} e^\alpha & 0 \\ 0 & e^\delta \end{bmatrix}.$$

$$e^A = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} e^\alpha & 0 \\ 0 & e^\delta \end{bmatrix} \Rightarrow \begin{aligned} e^\alpha &= -1 \\ e^\delta &= -4 \end{aligned} \lightning \quad \nexists$$

(4) Нека су $A, B \in M_n(\mathbb{R})$ и нека $\exists B^{-1}$. Доказати $e^{B^{-1}AB} = B^{-1}e^A B$.

$$e^{B^{-1}AB} = \sum_{k=0}^{\infty} \frac{(B^{-1}AB)^k}{k!} = \sum_{k=0}^{\infty} \frac{B^{-1}A^kB}{k!} = B^{-1}\cdot\left(\sum_{k=0}^{\infty} \frac{A^k}{k!}\right)\cdot B = B^{-1}\cdot e^{A}\cdot B.$$

$$(B^{-1}AB)^k = \overset{1}{(B^{-1}AB)}\overset{2}{(B^{-1}AB)}\cdots\overset{k}{(B^{-1}AB)} =$$

$$= B^{-1}\overset{1}{A}\underbrace{(BB^{-1})}_{=E}\overset{2}{A}\underbrace{(BB^{-1})}_{=E}\cdots\underbrace{(BB^{-1})}_{=E}\overset{k}{A}B =$$

$$= B^{-1}\overbrace{AA\cdots A}^{k}B = B^{-1}A^kB.$$