Линеарни системи ДЈ са константним коефицијентима (ЛСДЈКК)

$Y' = AY \quad (*)$, $A \in M_n(\mathbb{R})$

$Y, Y' \in \mathbb{R}^n$, $Y : \mathbb{R} \to \mathbb{R}^n$

$\varphi_1, \dots, \varphi_n$ – лин. нез. решења система $(*)$ = фундаментални скуп реш.

$\Phi = [\varphi_1 \downarrow \; \dots \; \varphi_n \downarrow] \in M_n(\dots)$ = фундаментална матрица

ОР: $Y(x) = \Phi(x) \cdot C = c_1\varphi_1 + c_2\varphi_2 + \dots + c_n\varphi_n$, $c_1, \dots, c_n \in \mathbb{R}$

$$C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$$

$W(\varphi_1, \dots, \varphi_n)(x) = W(x) = \det \Phi(x)$ – Вронскијан

$W(x) \neq 0 \Leftrightarrow \Phi$ фунд. матрица

Ојлеров метод

идеја: $\left.\begin{array}{l} \lambda \text{ – сопс. вр. } A \\ v \text{ – сопс. век. за } \lambda \end{array}\right\} Av = \lambda v$

$\varphi(x) = e^{\lambda x} \cdot v$ је реш $(*)$

$\varphi'(x) = A \cdot \varphi(x)$?

$\varphi'(x) = (e^{\lambda x} \cdot v)' = \lambda e^{\lambda x} \cdot v = e^{\lambda x} A v = A\varphi(x)$

① $y_1' = 5y_1 + 4y_2$ [случај реалних и различитих сопс. вр.]

$y_2' = 4y_1 + 5y_2$

$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $Y' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$

$E = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} = I = Id$

→ сопс. вр: $\det(A - \lambda E) = 0$

$$\det\left(\begin{bmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{bmatrix}\right) = 0$$

$$(5-\lambda)^2-4^2=0$$

$$(1-\lambda)(9-\lambda)=0 \longrightarrow \begin{matrix}\lambda_1=1\\ \lambda_2=9\end{matrix}$$

→ сопс. век: $\underline{\lambda_1=1}$ $\quad (A-\lambda_1 E)\gamma_1=0$

$$\begin{bmatrix}4&4\\4&4\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\vec{0} \qquad \begin{matrix}4a+4b=0 \Rightarrow b=-a\\ \cancel{4a+4b=0}\end{matrix}, \quad \begin{bmatrix}a\\-a\end{bmatrix}, \quad \gamma_1=\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\varphi_1(x)=e^{\lambda_1 x}\gamma_1=e^x\cdot\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}e^x\\-e^x\end{bmatrix}.$$

$\underline{\lambda_2=9}$ $\quad (A-\lambda_2 E)\gamma_2=0$

$$\begin{bmatrix}-4&4\\4&-4\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \qquad \begin{matrix}-4a+4b=0 \Rightarrow a=b\\ \cancel{4a-4b=0}\end{matrix}, \quad \begin{bmatrix}a\\a\end{bmatrix}, \quad \gamma_2=\begin{bmatrix}1\\1\end{bmatrix}$$

$$\varphi_2(x)=e^{\lambda_2 x}\gamma_2=e^{9x}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}e^{9x}\\e^{9x}\end{bmatrix}.$$

У овом случају су решења увек лин. нез. $(W(x)\neq 0)$, тј. ОР: $y(x)=c_1\varphi_1(x)+c_2\varphi_2(x)$, $c_1,c_2\in\mathbb{R}$

$$y(x)=c_1\begin{bmatrix}e^x\\-e^x\end{bmatrix}+c_2\begin{bmatrix}e^{9x}\\e^{9x}\end{bmatrix}=\begin{bmatrix}e^x&e^{9x}\\-e^x&e^{9x}\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$$

$$y_1(x)=c_1e^x+c_2e^{9x}$$
$$y_2(x)=-c_1e^x+c_2e^{9x}$$

(2) $y_1'=2y_1+y_2$ $\qquad$ [случај комплексних и различитих сопс. вр.]

$y_2'=y_1+3y_2-y_3$

$y_3'=-y_1+2y_2+3y_3$

$$y'=Ay, \quad A=\begin{bmatrix}2&1&0\\1&3&-1\\-1&2&3\end{bmatrix}$$

$$\det(A-\lambda E)=0$$

$$\begin{vmatrix}2-\lambda&1&0\\1&3-\lambda&-1\\-1&2&3-\lambda\end{vmatrix}=0$$

$$(2-\lambda)(3-\lambda)^2 + (-1)^2 + 0 - 2(-1)(2-\lambda) - (3-\lambda) = 0$$

$$(2-\lambda)(3-\lambda)^2 + \underbrace{1+4-2\lambda-3+\lambda}_{2-\lambda} = 0$$

$$D = 36 - 4\cdot 10 = -4$$

$$(2-\lambda)(10-6\lambda+\lambda^2) = 0$$

$$\lambda_1 = 2 \quad , \quad \lambda_{2/3} = \frac{6 \pm i 2}{2} = 3 \pm i \qquad (\lambda_3 = \overline{\lambda_2})$$

$\underline{\lambda_1 = 2}$

$$(A-\lambda_1 E)\gamma_1 = 0 \quad \dots \quad \gamma_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad , \quad \varphi_1(x) = e^{2x} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{2x} \\ 0 \\ e^{2x} \end{bmatrix}.$$

$\underline{\lambda_2 = 3+i}$

$(A-\lambda_2 E)\gamma_2 = 0$ → компл. сопс. век.

(решавамо у $\mathbb{C}$)

$$\begin{bmatrix} 2-(3+i) & 1 & 0 \\ 1 & 3-(3+i) & -1 \\ -1 & 2 & 3-(3+i) \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-1-i$, $-i$, $-i$

$a, b, c \in \mathbb{C}$

$$\begin{aligned} (-1-i)a + b &= 0 \;/\cdot i \;/\cdot(-2) \\ a - ib - c &= 0 \;+ \\ -a + 2b - ic &= 0 \;+ \end{aligned}$$

$(-1-i)\cdot i = -i+1$

$$\begin{aligned} (-1-i)a + b &= 0 \\ (2-i)a - c &= 0 \;/\cdot(-i) \\ (1+2i)a - ic &= 0 \;+ \end{aligned}$$

$$\begin{aligned} (-1-i)a + b &= 0 \\ (2-i)a - c &= 0 \\ \cancel{0 = 0} & \end{aligned}$$

$(2-i)(-i)+1+2i = -2i+i^2+1+2i = 0$

$$c = (2-i)a \quad , \quad \gamma_2 = \begin{bmatrix} a \\ (1+i)a \\ (2-i)a \end{bmatrix}$$
$$b = (1+i)a$$

$$\gamma_2 = \begin{bmatrix} 1 \\ 1+i \\ 2-i \end{bmatrix}$$

$\lambda_3 = \overline{\lambda_2} \rightsquigarrow \gamma_3 = \overline{\gamma_2}$ → нема потребе за λ_3 рачунати

$(A-\lambda_2 E)\gamma_2 = 0$

$(A-\lambda_3 E)\gamma_3 = 0$

$(\lambda_2, \gamma_2) \rightsquigarrow \varphi_2(x)$ и $\varphi_3(x)$

$$\varphi_2(x) = \mathrm{Re}\left(e^{\lambda_2 x}\gamma_2\right) \quad и \quad \varphi_3(x) = \mathrm{Im}\left(e^{\lambda_2 x}\gamma_2\right)$$

$$\psi(x) = e^{\lambda_2 x}\cdot \gamma_2 = e^{3x}\cdot(\cos x + i\sin x)\cdot \begin{bmatrix}1\\ 1+i\\ 2-i\end{bmatrix} = e^{3x}\cdot\begin{bmatrix}\cos x+i\sin x\\ \cos x + i\sin x + i\cos x - \sin x\\ 2\cos x+2i\sin x - i\cos x + \sin x\end{bmatrix}$$

$$\varphi_2(x) = e^{3x}\cdot\begin{bmatrix}\cos x\\ \cos x - \sin x\\ 2\cos x + \sin x\end{bmatrix}, \quad \varphi_3(x) = e^{3x}\cdot\begin{bmatrix}\sin x\\ \sin x + \cos x\\ 2\sin x - \cos x\end{bmatrix}$$

И овде важи $W(x) \neq 0 \Rightarrow$ ОР: $y(x) = c_1\varphi_1(x) + c_2\varphi_2(x) + c_3\varphi_3(x)$, $c_1, c_2, c_3 \in \mathbb{R}$.

[случај поновљених реалних сопс. вр.]

$\lambda \in \mathbb{R}$ сопс. вр. од A вишеструкости k

$n = \dim A$, $r = \operatorname{rang}(A-\lambda E)$, $m = n - r \rightarrow$ говори колико лин. нез. реш. система $(A-\lambda E)\gamma = 0$ имамо

1° $m = k$, $\gamma_1, \dots, \gamma_k$ решења од $(A-\lambda E)\gamma = 0 \rightsquigarrow \varphi_1(x) = e^{\lambda x}\gamma_1, \dots, \varphi_k(x) = e^{\lambda x}\gamma_k$

2° $m < k$, решење тражимо у облику $\varphi(x) = \boxed{P_{k-m}[x]}\cdot e^{\lambda x}$ → вектор полинома по x степена највише $k-m$

→ треба да нађемо k нез. реш.

(3) $y_1' = 2y_1 - y_2 - y_3$ [случај $m=k$]

$y_2' = 3y_1 - 2y_2 - 3y_3$

$y_3' = -y_1 + y_2 + 2y_3$

$$Y' = AY, \quad A = \begin{bmatrix}2 & -1 & -1\\ 3 & -2 & -3\\ -1 & 1 & 2\end{bmatrix}, \quad \det(A-\lambda E) = 0 \Rightarrow \lambda_1 = 0, \ \lambda_2 = \lambda_3 = 1$$

$\underline{\lambda_1 = 0}$ $\quad (A-\lambda_1 E)\gamma_1 = 0 \rightsquigarrow \gamma_1 = \begin{bmatrix}1\\ 3\\ -1\end{bmatrix}, \quad \varphi_1(x) = e^{0x}\cdot\gamma_1 = \begin{bmatrix}1\\ 3\\ -1\end{bmatrix}$

$\underline{\lambda_2 = \lambda_3 = 1}$

$k = 2$

$n = 3$

$$r = \operatorname{rang}(A-\lambda_2 E) = \operatorname{rang}\left(\begin{bmatrix}1 & -1 & -1\\ 3 & -3 & -3\\ -1 & 1 & 1\end{bmatrix}\right) = \operatorname{rang}\left(\begin{bmatrix}1 & -1 & -1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\right) = 1$$

$m = n - r = 3 - 1 = 2$

$k = m = 2 \rightarrow 1^\circ$

$$(A - \lambda_2 E)\gamma = 0$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 3 & -3 & -3 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{aligned} & a - b - c = 0 \Rightarrow c = a - b \\ & \cancel{3a - 3b - 3c = 0} \\ & \cancel{-a + b + c = 0} \end{aligned}$$

$$\begin{bmatrix} a \\ b \\ a-b \end{bmatrix} = a \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

$$\gamma_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \gamma_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\varphi_2(x) = e^{\lambda_2 x}\gamma_2 = \begin{bmatrix} e^x \\ 0 \\ e^x \end{bmatrix}, \quad \varphi_3(x) = e^{\lambda_2 x}\gamma_3 = \begin{bmatrix} 0 \\ e^x \\ -e^x \end{bmatrix}.$$

Независни? $\quad W(x) = \det\left(\begin{bmatrix} \varphi_1\downarrow & \varphi_2\downarrow & \varphi_3\downarrow \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & e^x & 0 \\ 3 & 0 & e^x \\ -1 & e^x & -e^x \end{bmatrix}\right) = -e^{2x} - e^{2x} + 3e^{2x} = e^{2x} \neq 0$

независни су $\Rightarrow$ ОР: $y(x) = \begin{bmatrix} 1 & e^x & 0 \\ 3 & 0 & e^x \\ -1 & e^x & -e^x \end{bmatrix} \cdot C, \; C \in \mathbb{R}^3$

④ $y' = Ay, \; A = \begin{bmatrix} 2 & -1 & -1 \\ 2 & -1 & -2 \\ -1 & 1 & 2 \end{bmatrix}$ $\quad$ [случај $m < k$]

$\det(A - \lambda E) = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 1 = \lambda$

$$\left.\begin{aligned} & k = 3 \\ & n = 3 \\ & r = \operatorname{rang}\left(\begin{bmatrix} 1 & -1 & -1 \\ 2 & -2 & -2 \\ -1 & 1 & 1 \end{bmatrix}\right) = \operatorname{rang}\left(\begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = 1 \end{aligned}\right\} m = n - r = 2 < 3 = k$$

Решења у облику: $\varphi(x) = P_{k-m}[x] \cdot e^{\lambda x} = e^x \cdot P_1[x] = e^x \cdot \begin{bmatrix} a_1 x + b_1 \\ a_2 x + b_2 \\ a_3 x + b_3 \end{bmatrix}$

$\varphi'(x) = A\varphi(x)$

$\left(e^x \cdot (ax + b)\right)' = e^x \cdot (ax + b) + e^x \cdot a$

$$\varphi'(x) = A\varphi(x)$$

$$\left(e^x\cdot(ax+b)\right)' = e^x\cdot(ax+b)+e^x\cdot a = e^x(ax+a+b)$$

$$\Rightarrow \cancel{e^x}\cdot \begin{bmatrix} a_1x+a_1+b_1 \\ a_2x+a_2+b_2 \\ a_3x+a_3+b_3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ 2 & -1 & -2 \\ -1 & 1 & 2 \end{bmatrix} \cancel{e^x}\cdot \begin{bmatrix} a_1x+b_1 \\ a_2x+b_2 \\ a_3x+b_3 \end{bmatrix}, \forall x \in \mathbb{R}$$

уз x:

$$a_1 = 2a_1 - a_2 - a_3 \Rightarrow a_1 - a_2 - a_3 = 0 \xrightarrow{\cdot(-1)} -a_1 + a_2 + a_3 = 0$$

$$\cancel{a_2 = 2a_1 - a_2 - 2a_3 \ (2)} \qquad \downarrow \cdot 2 \qquad \Downarrow$$

$$2a_1 - 2a_2 - 2a_3 = 0 \qquad a_3 = -a_1 + a_2 + 2a_3 \ (3)$$

$$\cancel{a_3 = -a_1 + a_2 + 2a_3 \ (3)} \qquad \Rightarrow a_2 = 2a_1 - a_2 - 2a_3 \ (2)$$

уз 1:

$$a_1 + b_1 = 2b_1 - b_2 - b_3$$

$$a_2 + b_2 = 2b_1 - b_2 - 2b_3$$

$$a_3 + b_3 = -b_1 + b_2 + 2b_3$$

$\cancel{6\times 6}$ 4×4

$$a_1 = 2a_1 - a_2 - a_3$$

$$\left.\begin{aligned} b_1 - b_2 - b_3 &= a_1 \\ 2b_1 - 2b_2 - 2b_3 &= a_2 \\ -b_1 + b_2 + b_3 &= a_3 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} 2a_1 &= a_2 \\ -a_1 &= a_3 \\ b_1 &= a_1 + b_2 + b_3 \end{aligned}\right\} a_1, b_2, b_3 \text{ – параметри}$$

$\cdot 2$, (-1)

(k параметара, k=3)

$$\varphi(x) = e^x\cdot\begin{bmatrix} a_1x + a_1 + b_2 + b_3 \\ 2a_1x + b_2 \\ -a_1x + b_3 \end{bmatrix} = a_1\cdot e^x\cdot\begin{bmatrix} x+1 \\ 2x \\ -x \end{bmatrix} + b_2\cdot e^x\cdot\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b_3\cdot e^x\cdot\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$a_1 = 1, b_2 = b_3 = 0$: $\varphi_1(x) = e^x\begin{bmatrix} x+1 \\ 2x \\ -x \end{bmatrix}$

$a_1 = b_3 = 0, b_2 = 1$: $\varphi_2(x) = e^x\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$a_1 = b_2 = 0, b_3 = 1$: $\varphi_3(x) = e^x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Независности?

$$W(x)=\det\left(\begin{bmatrix} e^{x}(x+1) & e^{x} & e^{x} \\ 2xe^{x} & e^{x} & 0 \\ -xe^{x} & 0 & e^{x} \end{bmatrix}\right)=e^{3x}(x+1)+e^{3x}\cdot x-2x e^{3x}=e^{3x}\neq 0$$

$\Rightarrow$ ОР: $y(x)=c_1\varphi_1(x)+c_2\varphi_2(x)+c_3\varphi_3(x)$, $c_1,c_2,c_3\in\mathbb{R}$

(*) Ако имамо $\lambda\in\mathbb{C}$ вишеструко, сличан је поступак, само на крају узимамо Re, Im   НЕЋЕМО РАДИТИ!

↳ у (3) и (4)