Парцијалне ДЈ 1. реда

$u(x) \leftarrow$ ОДЈ

$u(x_1, \dots, x_n) \leftarrow$ ПДЈ

- $f\left(x_1, \dots, x_n, u, \underbrace{\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}}_{1.\ \text{реда}}\right) = 0$, $n \geq 2$, општи облик ПДЈ 1. реда

- Квазилинеарне ПДЈ 1. реда:

(КВЛ)

$$\sum_{k=1}^{n} X_k(x_1, \dots, x_n, u) \cdot \frac{\partial u}{\partial x_k} = R(x_1, \dots, x_n, u)$$

$X_k, R : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^n, \mathbb{R}^{n+1}$

спец: $\sum_{k=1}^{n} X_k(x_1, \dots, x_n) \cdot \frac{\partial u}{\partial x_k} = 0$

хомогена линеарна (ХЛ)

- 1) КВЛ $\rightsquigarrow$ ХЛ

$$\sum X_k(x_1, \dots, x_n, u) \frac{\partial u}{\partial x_k} = R(x_1, \dots, x_n, u) \qquad n \text{ пром.}$$

$$\Big\} \; v = v(x_1, \dots, x_n, u)$$

$$\sum X_k(x_1, \dots, x_n, u) \cdot \frac{\partial v}{\partial x_k} + R(x_1, \dots, x_n, u) \cdot \frac{\partial v}{\partial u} = 0 \quad (\diamond) \qquad n+1 \text{ пром.}$$

- 2) ХЛ $\rightsquigarrow$ систем карактеристика

$$\sum X_k(x_1, \dots, x_n) \cdot \frac{\partial u}{\partial x_k} = 0 \qquad (\#)$$

$$\Downarrow$$

$$\frac{dx_1}{X_1(x_1, \dots, x_n)} = \frac{dx_2}{X_2(x_1, \dots, x_n)} = \dots = \frac{dx_n}{X_n(x_1, \dots, x_n)} \qquad (*)$$

- 3) Наћи независне прве интеграле система $(*)$: $\psi_1, \dots, \psi_{n-1}$

$\Rightarrow$ ОР од $(\#)$: $u = \varphi(\psi_1, \dots, \psi_{n-1})$, $\varphi \in C^1(\mathbb{R}^{n-1})$

ОР од $(\diamond)$: $\varphi(\psi_1, \dots, \psi_n) = 0$, $\varphi \in C^1(\mathbb{R}^n)$.

① $m, n, k \in \mathbb{R} \setminus \{0\}$. Решити ПДЈ:

$$(mz - ny)\cdot\frac{\partial u}{\partial x} + (nx - kz)\cdot\frac{\partial u}{\partial y} + (ky - mx)\cdot\frac{\partial u}{\partial z} = 0 \rightsquigarrow \text{ХЛ}$$

$$\frac{dx}{mz - ny} = \frac{dy}{nx - kz} = \frac{dz}{ky - mx}, \quad \psi_1, \psi_2 = ?$$

$$\frac{\alpha\, dx + \beta\, dy}{\alpha(mz - ny) + \beta(nx - kz)} = \frac{dz}{ky - mx}$$

уз z: $\alpha m - \beta k = 0$

$\alpha = k$
$\beta = m$
$$\Rightarrow \frac{k\,dx + m\,dy}{\cancel{-kny + nmx}_{-n}} = \frac{dz}{\cancel{ky - mx}} \Rightarrow k\,dx + m\,dy + n\,dz = 0 \;/\int$$

$$kx + my + nz = c_1 = \psi_1$$

$$\frac{\alpha x\, dx + \beta y\, dy}{\alpha x(mz - ny) + \beta y(nx - kz)} = \frac{z\,dz}{(ky - mx)\cdot z}$$

уз xy: $-\alpha n + \beta n = 0$

$\alpha = \beta = 1$:
$$\frac{x\,dx + y\,dy}{mxz - kyz} = \frac{z\,dz}{kyz - mxz} \Rightarrow x\,dx + y\,dy = -z\,dz \;/\int$$

$$x^2 + y^2 + z^2 = c_2 = \psi_2$$

ψ_1, ψ_2 нез.:

z нез. пром.
$ky - mx \neq 0$

$$\frac{D(\psi_1, \psi_2)}{D(x, y)} = \begin{vmatrix} k & m \\ 2x & 2y \end{vmatrix} = 2(ky - mx) \neq 0$$

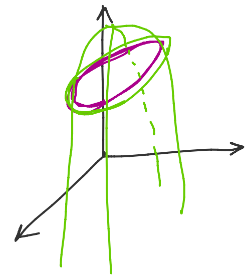
ОР: $u = \varphi(kx + my + nz, x^2 + y^2 + z^2), \quad \varphi \in C^1(\mathbb{R}^2)$

не појављује се x_k

• Кошијев проблем: Одредити решење које задовољава $u\big|_{x_k = t} = f(x_1, \ldots, \hat{x}_k, \ldots, x_n)$

$\rightarrow$ наћи решење које садржи дату „површ"
(крива, површ, 3Д ...)

$\varphi = ?$



$$\overline{\psi_k} = \psi_k(x_1, \dots, x_{k-1}, t, x_{k+1}, \dots, x_n)$$

$$u\big|_{x_k=t} = f = g(\overline{\psi_1}, \dots, \overline{\psi_{n-1}})$$

↳ наћи овакво g!

Кошијево решење: $u = g(\psi_1, \dots, \psi_{n-1})$

Ово је за ХЛ, аналогно је за КВЛ.

② Решити Кошијев проблем:

$$x(z^2-y^2)\cdot\frac{\partial u}{\partial x} + y(x^2+z^2)\cdot\frac{\partial u}{\partial y} - z(x^2+y^2)\cdot\frac{\partial u}{\partial z} = 0 \quad , \quad u\big|_{x=1} = u(1,y,z) = (y+z)^2$$

ХЛ $\rightsquigarrow$ $$\frac{dx}{x(z^2-y^2)} = \frac{dy}{y(x^2+z^2)} = \frac{dz}{-z(x^2+y^2)}$$

зад ② са прошлог двочаса

$$\left.\begin{aligned} \psi_1 &= x^2+y^2+z^2 \\ \psi_2 &= \frac{x}{yz} \end{aligned}\right\} \text{ОР}: u = \varphi\left(x^2+y^2+z^2, \frac{x}{yz}\right), \ \varphi \in C^1(\mathbb{R}^2).$$

Тражимо φ тд: $u(1,y,z) = (y+z)^2$

$$u(1,y,z) = \varphi\Big(\underbrace{1+y^2+z^2}_{\overline{\psi_1}}, \underbrace{\frac{1}{yz}}_{\overline{\psi_2}}\Big)$$

$$\overline{\psi_1} = \psi_1(1,y,z) = 1+y^2+z^2$$

$$\overline{\psi_2} = \psi_2(1,y,z) = \frac{1}{yz}$$

$g = ?$ $\quad g(\overline{\psi_1}, \overline{\psi_2}) = (y+z)^2$

$$g\left(1+y^2+z^2, \frac{1}{yz}\right) = (y+z)^2$$

$$(y+z)^2 = y^2+2yz+z^2 = (y^2+z^2+1) - 1 + \frac{2}{\frac{1}{yz}} = \overline{\psi_1} - 1 + \frac{2}{\overline{\psi_2}} = g(\overline{\psi_1}, \overline{\psi_2})$$

Кошијево решење: $u = g(\psi_1, \psi_2) = \psi_1 - 1 + \frac{2}{\psi_2} = x^2+y^2+z^2-1+\frac{2yz}{x}$.

③ Решити Кошијев проблем

$$x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} + z^2 = 0 \quad , \quad \begin{matrix} z=1 \\ xy = x+y \end{matrix}$$

$$\downarrow$$

$$x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = \underbrace{\boxed{-z^2}}_{R(x,y,z)}$$

КВЛ $\rightsquigarrow u(x,y,z)$
(ХЛ)

$$x^2\cdot\frac{\partial u}{\partial x} + y^2\cdot\frac{\partial u}{\partial y} - z^2\cdot\frac{\partial u}{\partial z} = 0$$

$$\Downarrow$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{-z^2}$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} \Big/ \int \qquad\qquad \frac{dy}{y^2} = -\frac{dz}{z^2} \Big/ \int$$

$$-\frac{1}{x} = -\frac{1}{y} + C_1 \qquad\qquad -\frac{1}{y} = \frac{1}{z} + C_2$$

$$\psi_1 = \frac{1}{y} - \frac{1}{x} \qquad\qquad \psi_2 = \frac{1}{y} + \frac{1}{z}$$

$$\frac{D(\psi_1,\psi_2)}{D(x,z)} = \begin{vmatrix} \frac{1}{x^2} & 0 \\ 0 & -\frac{1}{z^2} \end{vmatrix} = -\frac{1}{x^2z^2} \neq 0$$

није се тражило →

ОР ХЛ: $u = \varphi(\psi_1,\psi_2)$

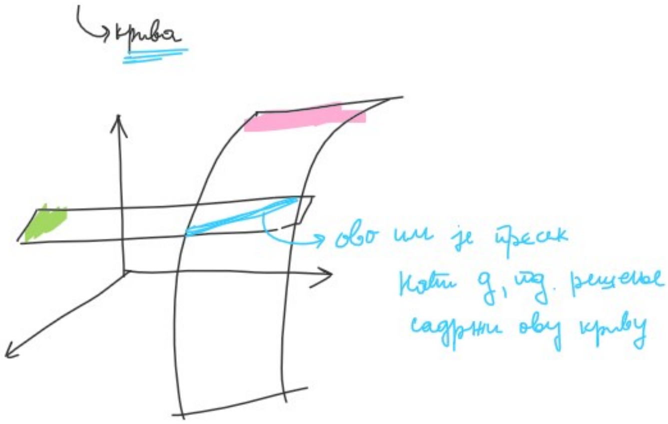
ОР КВЛ: $\varphi(\psi_1,\psi_2) = 0$ ← имплицитно задата фја $z(x,y)$

Кошијев проблем

$\begin{cases} z=1 \to \text{раван} \\ xy = x+y \to \text{површ} \end{cases}$
→ крива

$$\overline{\psi_1} = \psi_1\Big|_{z=1} = \frac{1}{y} - \frac{1}{x}$$

$$\overline{\psi_2} = \psi_2\Big|_{z=1} = \frac{1}{y} + 1$$



$$\overline{\psi_2} = \psi_2\big|_{z=1} = \frac{1}{y} + 1$$

Тражимо g тд. $g(\overline{\psi_1}, \overline{\psi_2}) = 0$, при условy $xy = x+y$.

$$xy = x+y \;/ : xy$$

$$1 = \frac{1}{x} + \frac{1}{y} = -\left(\frac{1}{y} - \frac{1}{x}\right) + 2\left(\frac{1}{y} + 1\right) - 2 = -\overline{\psi_1} + 2\overline{\psi_2} - 2$$

$$\underbrace{-\overline{\psi_1} + 2\overline{\psi_2} - 3}_{g(\overline{\psi_1}, \overline{\psi_2})} = 0$$

Кош. реш.:

$$0 = g(\psi_1, \psi_2) = -\psi_1 + 2\psi_2 - 3 = -\left(\frac{1}{y} - \frac{1}{x}\right) + 2\left(\frac{1}{y} + \frac{1}{z}\right) - 3$$

$$\frac{1}{x} + \frac{1}{y} + \frac{2}{z} = 3 \leftarrow \text{имп. задато } z(x,y)$$

Може чак и експлицитно: $\frac{2}{z} = 3 - \left(\frac{1}{x} + \frac{1}{y}\right)$

$$z = \frac{2}{3 - \left(\frac{1}{x} + \frac{1}{y}\right)}$$

II начин за Кошијево: параметризовати криву $z=1$, $xy = x+y$

$$y(x-1) = x$$

$$y = \frac{x}{x-1}$$

$$\gamma(t) = \left(t, \frac{t}{t-1}, 1\right)$$

$$\overline{\psi_1} = \psi_1\big|_{\gamma} = \psi_1\left(t, \frac{t}{t-1}, 1\right) = \frac{t-1}{t} - \frac{1}{t} = 1 - \frac{2}{t} \quad \Rightarrow \quad \frac{1}{t} = \frac{1}{2}\left(1 - \overline{\psi_1}\right)$$

$$\overline{\psi_2} = \psi_2\big|_{\gamma} = \psi_2\left(t, \frac{t}{t-1}, 1\right) = \frac{t-1}{t} + 1 = 2 - \frac{1}{t} \quad \Rightarrow \quad \frac{1}{t} = 2 - \overline{\psi_2}$$

$$\frac{1}{2}(1-\overline{\psi}_1)=2-\overline{\psi}_2 \Rightarrow -\overline{\psi}_1+2\overline{\psi}_2-3=0 \ldots$$

④ Решити Кошијев проблем

$$x(x^2+3y^2)\frac{\partial z}{\partial x}+y(3x^2+y^2)\frac{\partial z}{\partial y}=2z(x^2+y^2), \quad xy=z, \quad x^2-y^2=z^2$$

КВЛ $\rightsquigarrow \underline{u}(x,y,z)$

ХЛ: $x(x^2+3y^2)\cdot\frac{\partial u}{\partial x}+y(3x^2+y^2)\cdot\frac{\partial u}{\partial y}+2z(x^2+y^2)\cdot\frac{\partial u}{\partial z}=0$

$$\Downarrow$$

$$\frac{dx}{x(x^2+3y^2)}=\frac{dy}{y(3x^2+y^2)}=\frac{dz}{2z(x^2+y^2)}, \quad \psi_1,\psi_2=?$$

$$\frac{\frac{\alpha}{x}dx+\frac{\beta}{y}dy}{\alpha(x^2+3y^2)+\beta(3x^2+y^2)}=\frac{\frac{1}{z}dz}{2(x^2+y^2)}$$

$\to x^2(\alpha+3\beta)+y^2(3\alpha+\beta)$

$$\left.\begin{aligned}\alpha+3\beta&=2\\3\alpha+\beta&=2\end{aligned}\right\}$$

$$\alpha=\beta=\frac{1}{2}$$

$$\frac{dx}{2x}+\frac{dy}{2y}=\frac{dz}{z}\ /\int$$

$$\frac{1}{2}\ln|x|+\frac{1}{2}\ln|y|=\ln|z|+c_1\ /\cdot 2$$

$$\therefore\ \psi_1=\frac{yx}{z^2}$$

$$\frac{\alpha x\,dx+\beta y\,dy}{\alpha x^2(x^2+3y^2)+\beta y^2(y^2+3x^2)}=\frac{\frac{dz}{z}}{2(x^2+y^2)}$$

← Овакво делимо са z, јер се лево не појављује

← ово желимо да се скрати

$\to x^4(\alpha)+y^4(\beta)+x^2y^2(3\alpha+3\beta)$

идеја: ~~1)~~ направити квадрат бинома $(x^2+y^2)^2$

$\boxed{2)}$ направити разлику квадрата $x^4-y^4=(x^2-y^2)(x^2+y^2)$

идеја: ~~1)~~ направити квадрат бинома $(x+y)$

$\boxed{2)}$ направити разлику квадрата $x^4-y^4=(x^2-y^2)(x^2+y^2)$

(+ нешто ће нестане)

$\alpha=1$

$\beta=-1$

$$\frac{x\,dx-y\,dy}{\underbrace{x^4-y^4}_{(x^2-y^2)(x^2+y^2)}+x^2y^2\cdot(0)}=\frac{\frac{dz}{z}}{2(x^2+y^2)} \Rightarrow \frac{2x\,dx-2y\,dy}{x^2-y^2}=\frac{dz}{z} \quad \Big/\int \; ?$$

$$\Downarrow$$

$$\int d\ln|x^2-y^2|=\ln|z|+c_1$$

$$\ln|x^2-y^2| \; \ldots$$

$$d\left(\ln|x^2-y^2|\right)=\frac{1}{x^2-y^2}\cdot 2x\,dx+\frac{1}{x^2-y^2}\cdot(-2y)\,dy=\frac{2x\,dx-2y\,dy}{x^2-y^2}$$

$$\psi_2=\frac{x^2-y^2}{z}$$

ψ_1, ψ_2 нез ← добити

Кошијево реш? $\left.\begin{array}{l} xy=z \\ x^2-y^2=z^2 \end{array}\right\}$ у пресеку је крива C

$$\left.\begin{array}{l} \overline{\psi_1}=\psi_1\Big|_C=\left(\dfrac{xy}{z^2}\right)\Big|_C \underset{xy=z}{=} \dfrac{z}{z^2}=\dfrac{1}{z} \\[3em] \overline{\psi_2}=\psi_2\Big|_C=\left(\dfrac{x^2-y^2}{z}\right)\Big|_C \underset{x^2-y^2=z^2}{=} \dfrac{z^2}{z}=z \end{array}\right\} \quad \overline{\psi_1}=\frac{1}{\overline{\psi_2}} \Rightarrow \underbrace{\overline{\psi_1}-\frac{1}{\overline{\psi_2}}=0}_{g(\overline{\psi_1},\overline{\psi_2})}$$

КР: $g(\psi_1,\psi_2)=0$

$$\psi_1-\frac{1}{\psi_2}=0$$

$$\frac{xy}{z^2}-\frac{z}{x^2-y^2}=0 \Rightarrow z^3=xy(x^2-y^2).$$