① Свести систем ДЈ на систем у нормалном облику

$y_1' = f_1(x, y_1, \dots, y_n)$

$y_2' = f_2(x, y_1, \dots, y_n)$ 1. реда

$\vdots$

$y_n' = f_n(x, y_1, \dots, y_n)$

$$\boxed{y'''} = xy'' + y - z' + x + 1 \quad \rightarrow 3.\ \text{реда}$$

$$\boxed{z''} = y' \sin x + y - z - x \quad \rightarrow 2.\ \text{реда}$$

5 јна 1. реда

$\rightarrow y$

$$\boxed{\begin{aligned} &\rightarrow y' = y_1 \\ &\rightarrow y_2 = y_1' = y'' \end{aligned}}$$

$y''' = (y'')' = y_2'$

$\boxed{z_1 = z'}$

$z'' = (z')' = z_1'$

$$\left.\begin{aligned} y_2' &= xy_2 + y - z_1 + x + 1 \\ z_1' &= y_1 \sin x + y - z - x \\ y' &= y_1 \\ y_1' &= y_2 \\ z' &= z_1 \end{aligned}\right\} \text{норм. облику}$$

(y, y_1, y_2, z, z_1)

② Методом елиминације решити систем ДЈ:

а) $y' = py - qz$

$z' = qy + pz$

$p, q \in \mathbb{R} \setminus \{0\}$

б) $y_1' = y_2$

$y_2' = y_1$

$y_3' = y_1 + y_2 + y_3$

а) $y' = py - qz \Rightarrow z = \dfrac{py - y'}{q} \Rightarrow z' = \dfrac{py' - y''}{q}$

$z' = qy + pz \Rightarrow \dfrac{py' - y''}{q} = q \cdot y + p \cdot \dfrac{py - y'}{q} \; / \cdot q$

$y'' - 2py' + (p^2 + q^2)y = 0$ $\qquad \lambda^2 - 2p\lambda + (p^2 + q^2) = 0$

$$y'' - 2py' + (p^2+q^2)y = 0 \qquad \lambda^2 - 2p\lambda + (p^2+q^2) = 0$$

$$D = 4p^2 - 4(p^2+q^2) = -4q^2$$

$$\lambda_{1/2} = \frac{2p \pm i2q}{2} = p \pm iq$$

$$\underbrace{y(x) = c_1 e^{px}\cdot\cos(qx) + c_2 e^{px}\cdot\sin(qx)}, \quad c_1, c_2 \in \mathbb{R}$$

$$y' = c_1 e^{px}\left(p\cdot\cos qx - \sin qx \cdot q\right) + c_2 e^{px}\left(p\cdot\sin qx + \cos qx \cdot q\right)$$

$$\underbrace{z = \frac{p}{q}y - \frac{1}{q}y' = c_1 e^{px}\sin qx - c_2 e^{px}\cos qx}$$

б) $\left.\begin{array}{l} y_1' = y_2 \\ y_2' = y_1 \end{array}\right\} \longrightarrow y_2' = y_1''$

$$y_1'' = y_1 \overset{\lambda^2-1=0}{\Rightarrow} y_1 = c_1 e^{x} + c_2 e^{-x}, \quad c_1, c_2 \in \mathbb{R}$$

$$y_2 = y_1' = c_1 e^{x} - c_2 e^{-x}$$

$$y_3' = y_1 + y_2 + y_3 = 2c_1 e^{x} + y_3 \Rightarrow y_3' - y_3 = 2c_1 e^{x}$$

$p(x) = -1$

$q(x) = 2c_1 e^{x}$

$\int p(x)\,dx = -x$

$\int q(x)\cdot e^{\int p(x)dx}\,dx = \int 2c_1 e^{x}\cdot e^{-x}\,dx$

$= \int 2c_1\,dx = 2c_1 x$

$$\underbrace{y_3 = e^{-\int p\,dx}\left(c_3 + \int q\, e^{\int p\,dx}\,dx\right) = e^{x}\left(c_3 + 2c_1 x\right)}, \quad c_3 \in \mathbb{R}$$

③ Помоћу одговарајуће смене независне променљиве решити систем ДЈ:

а) $2\sqrt{x}\cdot y' = 2y - z$

$2\sqrt{x}\cdot z' = y + 2z$

б) $x y_1' = y_1 + y_2$

$x y_2' = y_2$

$x y_3' = -y_3$

$x y_2' = y_2 \Rightarrow x\dfrac{dy_2}{dx} = y_2$

$\Rightarrow \dfrac{dy_2}{y_2} = \dfrac{dx}{x} \;/\int$

$y_2 = cx$

$y_3 = \ldots$

$y_1 = \ldots$

$$Y(x) = \begin{bmatrix} y_1(x) \\ \vdots \\ y_n(x) \end{bmatrix}$$

$$Y' = F(x, Y) = (f_1, \ldots, f_n)$$

$$f(x)\cdot Y' = F(x, Y)$$

$f(x) = 2\sqrt{x},\ x,\ \ldots$

$x \rightsquigarrow t, \quad t(x) = ?, \quad x(t) = ?$

Знемимо:

$$f(x)\cdot\frac{dy}{dx} = \frac{dy}{dt} \Rightarrow \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{1}{f(x)}$$
$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$
$$\Rightarrow \frac{dt}{dx} = \frac{1}{f(x)}$$

а) $t(x) = ?$ $\quad , \underline{x \geq 0}$

$x=0$: $\left.\begin{array}{l} 2y(0) - z(0) = 0 \\ y(0) + 2z(0) = 0 \end{array}\right\} \Rightarrow y(0) = z(0) = 0$

$$\frac{dt}{dx} = \frac{1}{f(x)} = \frac{1}{2\sqrt{x}} \quad / \int \qquad (x \neq 0)$$

$$t(x) = \sqrt{x}, \quad t = \sqrt{x}, \quad x = t^2 \quad (t > 0)$$

$$\frac{dy}{dx} = \boxed{\frac{dy}{dt}}\cdot\frac{dt}{dx} = \frac{dy}{dt}\cdot\frac{d}{dx}(\sqrt{x}) = \frac{dy}{dt}\cdot\frac{1}{2\sqrt{x}} \Rightarrow \frac{dy}{dt} = \underbrace{2\sqrt{x}\cdot\frac{dy}{dx}}$$

$$- - - - \qquad \Rightarrow \frac{dz}{dt} = \underbrace{2\sqrt{x}\cdot\frac{dz}{dx}}$$

$$\Rightarrow \left.\begin{array}{l} \frac{dy}{dt} = y'_t = 2y - z \\ \frac{dz}{dt} = z'_t = y + 2z \end{array}\right\} \longrightarrow z = 2y - y'_t \rightarrow z' = 2y'_t - y''_t$$

$$2y'_t - y''_t = y + 2(2y - y'_t)$$

$$y''_t - 4y'_t + 5y = 0$$

$$\lambda^2 - 4\lambda + 5 = 0$$

$$D = 16 - 4\cdot 5 = -4$$

$$\lambda_{1/2} = 2 \pm i$$

$$y(t) = c_1 e^{2t}\cos t + c_2 e^{2t}\sin t, \quad c_1, c_2 \in \mathbb{R}$$

$$y'_t(t) = c_1 e^{2t}(-\sin t) + c_1 e^{2t}\cdot 2\cdot\cos t + c_2 e^{2t}\cos t + c_2 e^{2t}\cdot 2\cdot\sin t$$

$$z(t) = 2y(t) - y'_t(t) = c_1 e^{2t}\sin t - c_2 e^{2t}\cos t$$

$$y(x) = c_1 e^{2\sqrt{x}}\cos\sqrt{x} + c_2 e^{2\sqrt{x}}\sin\sqrt{x}$$
$$z(x) = c_1 e^{2\sqrt{x}}\sin\sqrt{x} - c_2 e^{2\sqrt{x}}\cos\sqrt{x}, \quad c_1, c_2 \in \mathbb{R}$$

б) $x \in \mathbb{R}$

$x=0$: $y_1(0) = y_2(0) = y_3(0) = 0.$

$f(x) = x \qquad (x \neq 0)$

$dt \quad 1 \_\_ = 1 /$

$f(x) = x$ $\quad (x \neq 0)$

$$\frac{dt}{dx} = \frac{1}{f(x)} = \frac{1}{x} \Big/ \int$$

$$t(x) = \ln|x|$$

1° $x > 0$

$$t(x) = \ln x \Rightarrow x = e^t$$

$$\frac{dy_1}{dt} = \frac{dy_1}{dx} \cdot \frac{dx}{dt} = \frac{dy_1}{dx} \cdot \frac{d}{dt}(e^t) = \frac{dy_1}{dx} \cdot e^t = \frac{dy_1}{dx} \cdot x$$

$$\frac{dy_2}{dt} = \ldots \quad \ldots \quad \ldots = \frac{dy_2}{dx} \cdot x$$

$$\frac{dy_3}{dt} = \ldots \quad \ldots \quad \ldots = \frac{dy_3}{dx} \cdot x$$

2° $x < 0$

$$t(x) = \ln(-x)$$

$\vdots$

Систем:

$$\frac{dy_1}{dt} = y_{1t}' = y_1 + y_2$$

$$\frac{dy_2}{dt} = y_{2t}' = y_2 \Rightarrow y_2(t) = c_2 e^t$$

$$\frac{dy_3}{dt} = y_{3t}' = -y_3 \Rightarrow y_3(t) = c_3 e^{-t}$$

$$y_{1t}' - y_1 = c_2 e^t$$

$$y_1(t) = e^t(c_1 + c_2 t)$$

$$y_1(x) = x(c_1 + c_2 \ln x)$$

$$y_2(x) = c_2 x \qquad c_1, c_2, c_3 \in \mathbb{R}$$

$$y_3(x) = \frac{c_3}{x}$$

(4) Методом елиминације решити систем ДЈ:

а) $y' = 1 - \frac{1}{z}$

$z' = \frac{1}{y - x}$

б) $2zy' = y^2 - z^2 + 1$

$z' = y + z$

а) $z' = \frac{1}{y-x} \Rightarrow y = x + \frac{1}{z'}$

$$\left.\begin{aligned} y' &= 1 - \frac{1}{(z')^2} \cdot z'' = 1 - \frac{z''}{(z')^2} \\ y' &= 1 - \frac{1}{z} \end{aligned}\right\} \quad 1 - \frac{z''}{(z')^2} = 1 - \frac{1}{z} \Rightarrow \frac{z''}{(z')^2} = \frac{1}{z}$$

$$z'' \cdot z = (z')^2$$ ← нелинеарна ДЈ 2. реда (нема x)

$$z'' \cdot z - (z')^2 = 0 \;/ \cdot z^2$$

$$z'' \cdot z - (z')^2 \ldots$$

$$(zz')' = z' \cdot z' + z \cdot z'' = z''z + (z')^2$$

$$(zz')' = z'\cdot z' + z\cdot z'' = z''z + (z')^2$$

$$\left(\frac{z'}{z}\right)' = \frac{z''\cdot z - (z')^2}{z^2}$$

$$z''\cdot z - (z')^2 = 0 \; / \cdot z^{-2}$$

$$\frac{z''\cdot z - (z')^2}{z^2} = 0$$

$$\left(\frac{z'}{z}\right)' = 0$$

$$\frac{z'}{z} = c_1 \; / \int$$

$$\vdots$$

$$z = c_2 e^{c_1 x}, \quad z' = c_2 c_1 e^{c_1 x}$$

$z \neq 0$

$y \neq x$ ✓

$z' \neq 0$

$$\boxed{y = x + \frac{1}{z'} = x + \frac{1}{c_1 c_2 e^{c_1 x}}, \quad c_1, c_2 \in \mathbb{R}\setminus\{0\}}$$

б) $y = z' - z$, $\quad y' = z'' - z'$

$$2z\cdot(z''-z') = (z'-z)^2 - z^2 + 1 = (z')^2 - \cancel{2zz'} + \cancel{z^2} - \cancel{z^2} + 1$$

$$\| $$

$$2zz'' - \cancel{2zz'}$$

$$\Rightarrow 2zz'' = (z')^2 + 1$$ ← нелин. 2. реда (нема x)

СМЕНА: $u(z) = z'$

$$u' = u'(z) = \frac{du}{dz} = \frac{d(z')}{dz} = \frac{d(z')}{dx}\cdot\frac{dx}{dz}$$

$$= z''\cdot\frac{dx}{dz}$$

$$\Rightarrow z'' = u'\cdot\frac{dz}{dx} = u'\cdot z' = u'u$$

$$u' = \frac{du}{dz}$$

$$2z\cdot u'u = u^2 + 1$$

$$\frac{2u\,du}{u^2+1} = \frac{dz}{z} \; / \int$$

$$\ln(u^2+1) = \ln|z|\cdot\tilde{c}_1$$

$$u^2 + 1 = c_1\cdot z, \quad c_1 \in \mathbb{R}\setminus\{0\}$$

$$u = \pm\sqrt{c_1 z - 1}, \quad c_1 z - 1 \geq 0$$

1° $u = \sqrt{c_1 z - 1}$, $\quad u = z'$

$$\frac{dz}{dx} = \sqrt{c_1 z - 1}$$

$$\frac{dz}{\sqrt{c_1 z - 1}} = dx \; / \int$$

$$\frac{2}{c_1}\sqrt{c_1 z - 1} = x + c_2 \Rightarrow c_1 z - 1 = \frac{c_1^2}{4}(x+c_2)^2 \Rightarrow z = \frac{1}{c_1} + \frac{c_1}{4}(x+c_2)^2$$

$$\Rightarrow z' = \frac{c_1}{2}(x+c_2)$$

$$y = z' - z = \frac{c_1}{2}(x+c_2) - \frac{1}{c_1} - \frac{c_1}{4}(x+c_2)^2$$

2° $u = -\sqrt{c_1 z - 1}$

$$\vdots$$