Решавање система методом првих интеграла

пр. $x_1' = x_2$

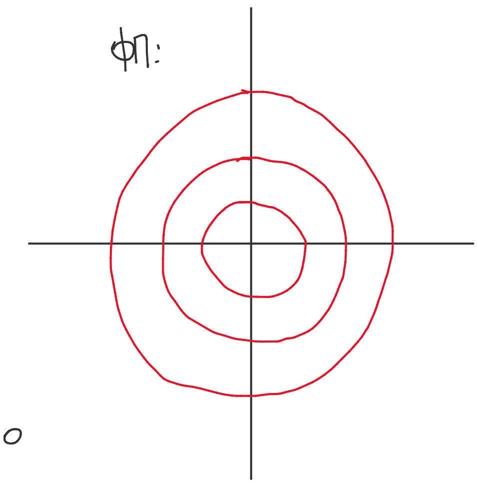
$x_2' = -x_1$

$$\begin{cases} x_1 = c_1\cos t + c_2 \sin t \\ x_2 = -c_1 \sin t + c_2 \cos t \end{cases}$$

експлицитно решење

имплицитно решење: $\underline{\underline{x_1^2 + x_2^2 = c}}$, $c > 0$

ФП:



$X \in \mathbb{R}^n$

$X' = F(X)$

ОР: (у имплицитном облику)

$\psi_1(x_1, \dots, x_n) = c_1$

$\psi_2(x_1, \dots, x_n) = c_2$

$\vdots$

$\psi_{n-1}(x_1, \dots, x_n) = c_{n-1}$

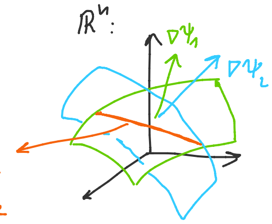
$\psi_1, \dots, \psi_{n-1}$ – први интеграли

и потребно је да буду независни

→ желимо да се хиперповрши секу трансверзално

$(M, \operatorname{codim} M = 1 = n - \dim M)$

$\mathbb{R}^n$:



$\psi_1 = c_1$ → одређује многострукост димензије $n-1$

$\psi_2 = c_2$ → –||–

пресек је $\dim = n-2$

(ако се секу трансверзално)

први интеграли независни $\iff$ сви $\psi_1 = c_1, \dots, \psi_{n-1} = c_{n-1}$ се секу трансверзално

$\iff$ $\nabla\psi_1, \dots, \nabla\psi_{n-1}$ у тачкама пресека независни

$$\frac{dx_1}{f_1(t, x_1, \dots, x_n)} = \frac{dx_2}{f_2(t, x_1, \dots, x_n)} = \dots = \frac{dx_n}{f_n(t, x_1, \dots, x_n)} = \frac{dt}{f_{n+1}(t, x_1, \dots, x_n)}$$

систем у симетричном облику

$\Updownarrow$

$$\frac{dx_1}{dt} = x_1' = \frac{f_1}{f_{n+1}}$$

$$x_2' = \frac{f_2}{f_{n+1}}$$

$$\vdots$$

$$x_n' = \frac{f_n}{f_{n+1}}$$

← систем у норм. облику

$$\vdots$$
$$x_n' = \frac{f_n}{f_{n+1}}$$

① $x' = \dfrac{x+y^2+t^2}{t}$ $\qquad \left[\ \dfrac{dx}{x+y+t^2} = \dfrac{dy}{y} = \dfrac{dt}{t}\ \right]$

$y' = \dfrac{y}{t}$

$$\frac{dy}{dt} = \frac{y}{t} \Rightarrow \frac{dy}{y} = \frac{dt}{t} \Rightarrow \ldots\ y = \underline{\underline{c_1}} \cdot t \Rightarrow c_1 = \frac{y}{t}, \quad \psi_1(t,x,y) = \frac{y}{t}$$

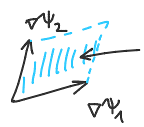
$$x' = \frac{x + t^2 \cdot c_1^2 + t^2}{t} = \frac{x}{t} + t\cdot(1+c_1^2) \overset{\text{лин}}{\rightsquigarrow} x(t) = c_2 t + (1+c_1^2)t^2$$

$$x = c_2 t + t^2 + t^2 \cdot \frac{y^2}{t^2} = c_2 t + t^2 + y^2$$

$$\Rightarrow c_2 = \frac{x - t^2 - y^2}{t}, \quad \psi_2(t,x,y) = \frac{x-t^2-y^2}{t}$$

$$\nabla\psi_1 = \left(-\frac{y}{t^2}, 0, \frac{1}{t}\right)$$

$$\nabla\psi_2 = \left(-\frac{x-y^2}{t^2} - 1, \frac{1}{t}, \frac{-2y}{t}\right)$$



$\|\nabla\psi_1 \times \nabla\psi_2\| \neq 0$

$$\nabla\psi_1 \times \nabla\psi_2 = \begin{vmatrix} -\frac{y}{t^2} & 0 & \frac{1}{t} \\ -\frac{x-y^2}{t^2}-1 & \frac{1}{t} & \frac{-2y}{t} \\ \vec{i} & \vec{j} & \vec{k} \end{vmatrix} = \underbrace{-\frac{1}{t^2}}_{\neq 0}\vec{i} + \frac{x+y^2+t^2}{t^3}\vec{j} - \frac{y}{t^3}\vec{k} \Rightarrow \text{независни су}$$

ОР: $\dfrac{y}{t} = c_1$

$\dfrac{x-t^2-y^2}{t} = c_2$, $\quad c_1, c_2 \in \mathbb{R}^2$

② $x' = \dfrac{4y-3t}{2y-3x}$

$y' = \dfrac{4x-2t}{2y-3x}$ $\qquad$ идеја: $\alpha x' + \beta y' = \dfrac{\alpha(4y-3t)+\beta(4x-2t)}{2y-3x}$

$$\alpha x' + \beta y' = \frac{\alpha(4y-3t)+\beta(4x-2t)}{2y-3x} = \frac{x(4\beta)+y(4\alpha)+t(-3\alpha-2\beta)}{2y-3x}$$

бирамо α и β: $\quad -3\alpha - 2\beta = 0$

нпр. $\alpha = 2,\ \beta = -3$

бирамо α и β: $-3\alpha-2\beta=0$

нпр. $\alpha=2,\ \beta=-3$

$$2x'-3y'=\frac{-12x+8y}{2y-3x}=4$$

$$2\frac{dx}{dt}-3\frac{dy}{dt}=4\ \Big/\int dt$$

$$2x-3y=4t+c_1 \rightsquigarrow \psi_1=2x-3y-4t$$

$$\alpha x\cdot x'+\beta y\cdot y'=\frac{\alpha x(4y-3t)+\beta y(4x-2t)}{2y-3x}=\frac{xy(4\alpha+4\beta)+xt(-3\alpha)+yt(-2\beta)}{2y-3x}$$

бирамо α и β тд. $4\alpha+4\beta=0$, нпр. $\alpha=1,\ \beta=-1$:

$\left(\frac{x^2}{2}\right)'=xx'$

$\left(\frac{y^2}{2}\right)'=yy'$

$$xx'-yy'=\frac{-3xt+2yt}{2y-3x}=t\ \Big/\int dt$$

$$\frac{x^2}{2}-\frac{y^2}{2}=\frac{t^2}{2}+c_2 \rightsquigarrow \psi_2=x^2-y^2-t^2$$

$$\nabla\psi_1=(-4,2,-3)$$

$$\nabla\psi_2=(-2t,2x,-2y)$$

$$\nabla\psi_1\times\nabla\psi_2=\begin{vmatrix}-4 & 2 & -3\\ -2t & 2x & -2y\\ \vec{i} & \vec{j} & \vec{k}\end{vmatrix}=(-4y+6x)\vec{i}+(-8y+6t)\vec{j}+(-8x+4t)\vec{k}$$

$$\nabla\psi_1\times\nabla\psi_2=\vec{0}\iff -4y+6x=-8y+6t=-8x+4t=0$$

$$3x-2y=-4y+3t=-2x+t=0$$

$(2x, x, \frac{3}{2}x) \leftarrow$ проблем

$$t=2x$$

$$3x-2y=0 \Rightarrow y=\frac{3}{2}x$$

$$\psi_1(2x,x,\tfrac{3}{2}x)=-8x+2x-\frac{9}{2}x=-\frac{21}{2}x\quad=c_1$$

$$\psi_2(2x,x,\tfrac{3}{2}x)=x^2-\frac{9}{4}x^2-4x^2=-\frac{21}{2}x^2\quad=c_2$$

$$\frac{c_1^2}{-\frac{21}{2}}=c_2 \Rightarrow c_1^2=-\frac{21}{2}c_2$$

ово не сме

$$\text{ОР}: \ \psi_1 = c_1 \qquad c_1^2 \neq -\frac{21}{2} c_2$$
$$\psi_2 = c_2$$

(3) $$x' = \frac{x(t^2+y^2)}{t(y^2-x^2)}$$

$$y' = \frac{y(t^2+x^2)}{t(x^2-y^2)}$$

$$\alpha x x' + \beta y y' = \frac{\alpha x^2(t^2+y^2) - \beta y^2(t^2+x^2)}{t(y^2-x^2)} = \frac{x^2y^2(\alpha-\beta) + x^2t^2(\alpha) + y^2t^2(-\beta)}{t(y^2-x^2)}$$

$$\alpha = \beta = 1:$$

$$xx' + yy' = \frac{x^2t^2 - y^2t^2}{t(y^2-x^2)} = \frac{t^2(x^2-y^2)}{t(y^2-x^2)} = -t \ \Big/ \int dt$$

$$\psi_1 = x^2 + y^2 + t^2$$

$$\frac{\alpha}{x}x' + \frac{\beta}{y}y' = \frac{\alpha(t^2+y^2) - \beta(t^2+x^2)}{t(y^2-x^2)} = \frac{t^2(\alpha-\beta) + x^2(-\beta) + y^2(\alpha)}{t(y^2-x^2)}$$

$$\alpha = \beta = 1:$$

$$(\ln|x|)' = \frac{x'}{x}$$

$$\frac{x'}{x} + \frac{y'}{y} = \frac{y^2-x^2}{t(y^2-x^2)} = \frac{1}{t} \ \Big/ \int dt$$

$$\ln|x| + \ln|y| = \ln|t| + c_2$$

$$\ln\left|\frac{xy}{t}\right| = c_2 \Rightarrow \frac{xy}{t} = \psi_2$$

проверить нез!

(4) $$x' = \frac{x^2-y^2-t^2}{2xt}$$

$$y' = \frac{y}{t}$$

$$\frac{dy}{dt} = y' = \frac{y}{t} \Rightarrow \frac{dy}{y} = \frac{dt}{t} \Rightarrow \ldots \ y = c_1 \cdot t \ , \ \psi_1 = \frac{y}{t}$$

$$x' = \frac{x^2 - c_1^2 t^2 - t^2}{2xt} = \frac{x}{2t} - \frac{(c_1^2+1)t}{2x} \qquad (\text{Бернулијева } \alpha = -1)$$

$$u = x^2$$

$$\frac{u'}{2} = \frac{u}{2t} - \frac{(c_1^2+1)t}{2}$$

$$u' = \frac{u}{t} - (c_1^2+1)t \rightsquigarrow u(t) = c_2 t - t^2(1+c_1^2) = c_2 t - t^2 - y^2$$

$$x^2 = c_2 t - t^2 - y^2$$

$$c_2 = \frac{x^2+t^2+y^2}{t} = \psi_2$$

$$x^2 = c_2 t - t(1+c_1^2)$$

$$x = \pm\sqrt{c_2 t - t(1+c_1^2)}$$

проверити нез!