Још мало о системима

(1) Помоћу одговарајуће смене независне променљиве решити систем ДЈ:

а) $2\sqrt{t}\cdot x_1' = 2x_1 - x_2$
$2\sqrt{t}\cdot x_2' = x_1 + 2x_2$

б) $t\cdot x_1' = x_1 + x_2$
$t\cdot x_2' = x_2$
$t\cdot x_3' = x_3$

$$f(t)\cdot X' = AX \xrightarrow{t\to\tau} X' = AX$$

знемо: $\boxed{f(t)}\cdot\dfrac{dx}{dt} = \dfrac{dx}{d\tau}$, $\quad \dfrac{dx}{d\tau} = \dfrac{dx}{dt}\cdot\boxed{\dfrac{dt}{d\tau}}$ $\Big\}\ f(t) = \dfrac{dt}{d\tau} \Rightarrow \dfrac{d\tau}{dt} = \dfrac{1}{f(t)} \Rightarrow \boxed{\tau = \int\dfrac{dt}{f(t)}}$

а) $f(t) = 2\sqrt{t}$, $t\geq 0$, $t>0$

$$\frac{d\tau}{dt} = \frac{1}{f(t)} = \frac{1}{2\sqrt{t}} \Rightarrow \tau = \int\frac{dt}{2\sqrt{t}} = \sqrt{t} + c$$

$$X(t) \rightsquigarrow X(\tau) = X(\sqrt{t}) \quad , \tau = \sqrt{t} \Rightarrow t = \tau^2$$

$$2\sqrt{t}\cdot\frac{dX}{dt} = 2\sqrt{t}\cdot\frac{dX}{d\tau}\cdot\frac{d\tau}{dt} = 2\sqrt{t}\cdot\frac{dX}{d\tau}\cdot\frac{1}{2\sqrt{t}} = \frac{dX}{d\tau}$$

$$\Rightarrow 2\sqrt{t}\cdot X'(t) = A\cdot X(t) \rightsquigarrow X'(\tau) = A\cdot X(\tau)$$

$$x_1' = 2x_1 - x_2$$
$$x_2' = x_1 + 2x_2$$

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \quad \begin{vmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)^2 + 1 = 0$$

$$2-\lambda_{1/2} = \pm i \Rightarrow \lambda_{1/2} = 2\pm i \rightsquigarrow J = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

$\lambda_1 = 2+i$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -ia - b = 0 \\ a - ib = 0 \end{matrix} \Rightarrow \gamma_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\cdot\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\text{ОР}: \ X(\tau) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\cdot e^{2\tau}\cdot R_{(\tau)}\cdot c, \ c\in\mathbb{R}^2$$

$$X(t) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot e^{2\sqrt{t}} \cdot \begin{bmatrix} \cos\sqrt{t} & \sin\sqrt{t} \\ -\sin\sqrt{t} & \cos\sqrt{t} \end{bmatrix} \cdot c \quad , c \in \mathbb{R}^2$$

б) $\frac{d\tau}{dt} = \frac{1}{t}$

$\tau(t) = \ln|t|$ → $\ln(t)$, $t > 0$; → $\ln(-t)$, $t < 0$

$t = 0: \; x_1(0) + x_2(0) = 0 = x_2(0) = x_3(0) \Rightarrow x_1(0) = x_2(0) = x_3(0) = 0.$

$\vdots$

② Решити систем:

$$t^2 x_1 x_1' = \frac{x_1^2}{2} + e^{2x_2}$$

$$2t^2 x_2' = -3\frac{x_1^2}{e^{2x_2}} - 4$$

$x_1^2 = y_1$

$y_1' = 2x_1 x_1'$

$t^2 \cdot \frac{y_1'}{2} = \frac{y_1}{2} + e^{2x_2}$

$t^2 \cdot 2x_2' = -3\frac{y_1}{e^{2x_2}} - 4 \,/\cdot e^{2x_2} \Rightarrow t^2 \cdot 2e^{2x_2} \cdot x_2' = -3y_1 - 4e^{2x_2}$

$y_2 = e^{2x_2}$

$y_2' = 2e^{2x_2} x_2'$

$t^2 \cdot \frac{y_1'}{2} = \frac{y_1}{2} + y_2 \quad /\cdot 2$

$t^2 \cdot y_2' = -3y_1 - 4y_2$

$\frac{d\tau}{dt} = \frac{1}{t^2}$

$\tau = -\frac{1}{t}$

$y_1'(\tau) = y_1(\tau) + 2y_2(\tau)$

$y_2'(\tau) = -3y_1(\tau) - 4y_2(\tau)$

$$y(\tau) = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} e^{-\tau} & 0 \\ 0 & e^{-2\tau} \end{bmatrix} \cdot c, \quad c \in \mathbb{R}^2$$

нпр. $y_1(\tau) = e^{-\tau} \cdot c_1 - 2 \cdot e^{-2\tau} \cdot c_2$

$y_1(t) = e^{1/t} \cdot c_1 - 2e^{2/t} \cdot c_2 > 0$

$x_1(t) = \pm\sqrt{c_1 e^{1/t} - 2c_2 e^{2/t}}$ итд.

③ $A \in M_2(\mathbb{R})$

$B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

а) Одредити једну фундаменталну матрицу система $X' = B^{-1}ABX$.

б) Одредити $B^{-1}AB$.

$e^{tA} = \begin{bmatrix} e^t & 0 \end{bmatrix}$

$B = \begin{bmatrix} \cdot & \cdot \\ 0 & 1 \end{bmatrix}$

$$e^{tA} = \begin{bmatrix} e^t & 0 \\ -3e^t + 3e^{2t} & e^{2t} \end{bmatrix}$$

б) Одредити $B^{-1}AB$.

а) $$\Phi(t) = e^{t(B^{-1}AB)} = B^{-1}e^{tA}\cdot B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} e^t & 0 \\ -3e^t+3e^{2t} & e^{2t} \end{bmatrix}\cdot\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7e^t - 6e^{2t} & 14e^t - 14e^{2t} \\ -3e^t + 3e^{2t} & -6e^t + 7e^{2t} \end{bmatrix}.$$

б) $C = B^{-1}AB$

$X' = CX \rightsquigarrow \Phi(t)$

I начин: $e^t, e^{2t} \rightsquigarrow C \sim \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$$\Rightarrow \exists S \in GL_2(\mathbb{R}), \quad C = S\cdot\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\cdot S^{-1}$$

$$\Phi(t) = e^{tC} = S\cdot e^{t\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}\cdot S^{-1} = S\cdot\begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}\cdot S^{-1} = \dots$$

$$S = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \Rightarrow S^{-1} = \frac{1}{\alpha\delta - \beta\gamma}\cdot\begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}$$

II начин: $\Phi(t)$ фунд, $\quad \Phi'(t) = C\cdot\Phi(t) \Rightarrow C = \Phi'(t)\cdot\Phi^{-1}(t) = \dots$

$$\Phi^{-1}(t) = (e^{tC})^{-1} = e^{(-t)C} = \Phi(-t) = \dots$$

решење: $\quad C = \begin{bmatrix} -5 & -14 \\ 3 & 8 \end{bmatrix}.$

(4) Методом елиминације решити системе:

а) $x_1' = x_2$
$x_2' = x_1 + x_2$

б) $x_1' = 1 - \frac{1}{x_2}$
$x_2' = \frac{1}{x_1 - t}$

в) $2x_2x_1' = x_1^2 - x_2^2 + 1$
$x_2' = x_1 + x_2$

а) $x_2 = x_1'$
$\Downarrow \quad x_1' = x_1 + x_2$
$x_1'' = x_1 + x_1'$
$x_1'' - x_1' - x_1 = 0$
$\vdots$

$x_1 = x_2' - x_2$
$\Downarrow \quad x_1' = x_2$
$x_2'' - x_2' - x_2 = 0$
$\vdots$

б) деф: $x_2 \neq 0$, $x_1 \neq t$

$$x_2' = \frac{1}{x_1 - t} \Rightarrow \frac{1}{x_2'} = x_1 - t \Rightarrow x_1 = \frac{1}{x_2'} + t \ /'$$

$$x_1' = -\frac{1}{(x_2')^2} \cdot x_2'' + 1$$

$$\Downarrow \ x_1' = 1 - \frac{1}{x_2}$$

$$-\frac{x_2''}{(x_2')^2} + 1 = 1 - \frac{1}{x_2}$$

$$\left(\frac{x_2'}{x_2}\right)' = \frac{x_2'' x_2 - (x_2')^2}{x_2^2}$$

$$\frac{x_2''}{(x_2')^2} = \frac{1}{x_2} \Rightarrow x_2'' x_2 = (x_2')^2$$

$$x_2'' x_2 - (x_2')^2 = 0 \ / : x_2^2$$

$$\left(\frac{x_2'}{x_2}\right)' = 0 \ / \int$$

$$\frac{x_2'}{x_2} = c_1 \ / \int$$

$$x_2 = c_2 \cdot e^{c_1 t}, \quad c_2 \neq 0, \ c_1 \neq 0$$

$$x_1 = t + \frac{1}{c_1 c_2 e^{c_1 t}}$$

в) $x_2' = x_1 + x_2 \Rightarrow x_1 = x_2' - x_2$

$$\Downarrow \ 2x_2 x_1' = x_1^2 - x_2^2 + 1$$

$$2x_2(x_2'' - x_2') = (x_2' - x_2)^2 - x_2^2 + 1$$

$$2x_2 x_2'' - \cancel{2x_2 x_2'} = (x_2')^2 - \cancel{2x_2' x_2} + \cancel{x_2^2} - \cancel{x_2^2} + 1$$

$$2x_2 x_2'' = (x_2')^2 + 1 \leftarrow \text{нелин. јна 2. реда (нема } t \text{ у себи)}$$

СМЕНА: $u(x_2) = x_2'$

$$u'(x_2) = \frac{du}{dx_2} = \frac{d(x_2')}{dx_2} = \underbrace{\frac{d(x_2')}{dt}}_{x_2''} \cdot \frac{dt}{dx_2}$$

$$\Rightarrow x_2'' = u' \cdot \frac{dx_2}{dt} = u' \cdot x_2' = u u'$$

$$2x_2 \cdot u u' = u^2 + 1$$

$$\frac{2uu'}{u^2+1} = \frac{1}{x_2}$$

$$\frac{2u\,du}{u^2+1} = \frac{dx_2}{x_2} \ / \int$$

$$\ln(u^2+1) = \ln|x_2| + \tilde{c}$$

$$\vdots$$

$$(x_2')^2 + 1 = c_1 \cdot x_2, \quad c_1 \in \mathbb{R} \setminus \{0\}$$

$\vdots$

$$(x_2')^2+1 = c_1 \cdot x_2 \quad , \quad c_1 \in \mathbb{R} \setminus \{0\}$$

$$x_2' = \pm\sqrt{c_1 x_2 - 1} \quad , \quad c_1 x_2 - 1 \geq 0$$

$$x_2' = -\sqrt{c_1 x_2 - 1}$$

$\vdots$

$$\frac{dx_2}{\sqrt{c_1 x_2 - 1}} = + \, dt \,/\!\int$$

$\vdots$

$$x_2 = \frac{1}{c_1}\left(1 + \frac{c_1^2}{4}(t+c_2)^2\right)$$

$$\Rightarrow x_1 = x_2' - x_2 = \ldots = \frac{c_1}{2}(t+c_2) - \frac{1}{c_1} - \frac{c_1}{4}(t+c_2)^2$$