Ојлерова ДЈ

$(at+b)^n \cdot x^{(n)} + p_1(at+b)^{n-1} x^{(n-1)} + \dots + p_{n-1}(at+b)x' + p_n x = 0$ ЛДЈВР са функционалним коеф.

$p_1, \dots, p_n, a, b \in \mathbb{R}$

често $b=0$, $a=1$

СМЕНА: $u = \ln|t| = \begin{cases} \ln(t), & t>0 \\ \ln(-t), & t<0 \end{cases}$ (тј. $u = \ln|at+b|$) $\rightsquigarrow$ своди се на ЛДЈВРКК

① $t^2 x'' - tx' + 4x = \cos(\ln t) + t \cdot \sin(\ln t)$, $t>0$

јесте Ојлерова $b=0, a=1$

$u = \ln(t) \Rightarrow t = e^u$

$$x' = \frac{dx}{dt} = \frac{dx}{du} \cdot \frac{du}{dt} = x'_u \cdot \frac{1}{t} = x'_u \cdot e^{-u}$$

$$x'' = \frac{d}{dt}\left(x'_u \cdot \frac{1}{t}\right) = \frac{d}{dt}(x'_u) \cdot \frac{1}{t} + x'_u \cdot \left(-\frac{1}{t^2}\right) = \frac{d(x'_u)}{du} \cdot \frac{du}{dt} \cdot \frac{1}{t} - \frac{1}{t^2} \cdot x'_u = x''_u \cdot \frac{1}{t^2} - \frac{1}{t^2} \cdot x'_u = \frac{1}{t^2}(x''_u - x'_u) = e^{-2u}(x''_u - x'_u)$$

$\Rightarrow t \cdot x' = x'_u$

$t^2 \cdot x'' = x''_u - x'_u$

$x''_u - x'_u - x'_u + 4x = \cos(u) + e^u \sin(u)$

$x''_u - 2x'_u + 4x = \cos u + e^u \sin u \longrightarrow$ нехом. ДЈВРКК

ХОМ: $\lambda^2 - 2\lambda + 4 = 0$

$\vdots$

$1 \pm i\sqrt{3} \rightsquigarrow e^u \cos(\sqrt{3}u)$

$e^u \sin(\sqrt{3}u)$

НЕХОМ: $g_1(u) = \cos u$

$\vdots$

$x_{p_1}(u) = a\cos u + b\sin u$

$\vdots$

$x_{p_1}(u) = \frac{3}{13}\cos u - \frac{2}{13}\sin u$

$g_2(u) = e^u \sin u$

$\vdots$

$x_{p_2}(u) = e^u(a\cos u + b\sin u)$

$\vdots$

$x_{p_2}(u) = \frac{e^u}{2}\sin u$

ОР: $x(u) = c_1 e^u \cos(\sqrt{3}u) + c_2 e^u \sin(\sqrt{3}u) + \frac{3}{13}\cos u - \frac{2}{13}\sin u + \frac{e^u}{2}\sin u$

$\Updownarrow$

$x(t) = c_1 t \cdot \cos(\sqrt{3}\ln t) + c_2 t \sin(\sqrt{3}\ln t) + \frac{3}{13}\cos(\ln t) - \frac{2}{13}\sin(\ln t) + \frac{t}{2}\sin(\ln t)$, $t>0$, $c_1, c_2 \in \mathbb{R}$.

Метода варијације константе (МВК)

$$x^{(n)}+a_1(t)x^{(n-1)}+\dots+a_{n-1}(t)\cdot x'+a_n(t)\cdot x=f(t)$$

$\varphi_1(t),\dots,\varphi_n(t)\to$ фунд. скуп решења (база ВП решења хомогене)

$$x_H(t)=c_1\varphi_1(t)+\dots c_n\varphi_n(t)$$

$$c_k \overset{\text{МВК}}{\rightsquigarrow} c_k(t)$$

Њихови изводи задовољавају: (Систем алгебарских јна.)

$$c_1'(t)\cdot\varphi_1(t)+c_2'(t)\varphi_2(t)+\dots+c_n'(t)\varphi_n(t)=0$$

$$c_1'(t)\cdot\varphi_1'(t)+\quad\dots\quad+c_n'(t)\cdot\varphi_n'(t)=0$$

$$\vdots$$

$$c_1'(t)\cdot\varphi_1^{(n-2)}(t)+\dots+c_n'(t)\cdot\varphi_n^{(n-2)}(t)=0$$

$$c_1'(t)\cdot\varphi_1^{(n-1)}(t)+\dots+c_n'(t)\cdot\varphi_n^{(n-1)}(t)=f(t)$$

$$\Phi(t)\cdot\begin{bmatrix}c_1'\\ \vdots\\ c_n'\end{bmatrix}=\begin{bmatrix}0\\ \vdots\\ 0\\ f\end{bmatrix}$$

$$\begin{bmatrix}c_1\\ \vdots\\ c_n\end{bmatrix}=\int\Phi^{-1}(t)\cdot\begin{bmatrix}0\\ \vdots\\ 0\\ f(t)\end{bmatrix}dt$$

$\varphi_k:\mathbb{R}\to\mathbb{R}$

$$W(\varphi_1,\dots,\varphi_n)(t)=\begin{vmatrix}\varphi_1&\varphi_2&\dots&\varphi_n\\ \varphi_1'&\varphi_2'&\dots&\varphi_n'\\ \vdots&\vdots&\ddots&\vdots\\ \varphi_1^{(n-1)}&\varphi_2^{(n-1)}&\dots&\varphi_n^{(n-1)}\end{vmatrix}=\det\Phi(t)$$

(Вронскијан)

$$\Phi(t)=\begin{bmatrix}\varphi_1&\varphi_2&\dots&\varphi_n\\ \varphi_1'&\varphi_2'&\dots&\varphi_n'\\ \vdots&\vdots&\ddots&\vdots\\ \varphi_1^{(n-1)}&\varphi_2^{(n-1)}&\dots&\varphi_n^{(n-1)}\end{bmatrix}$$

$\varphi_1,\dots,\varphi_n$ лин. нез. $\Leftrightarrow$ фунд. систем $\Leftrightarrow W(t)\neq0$

② $x''+4x=2\,\mathrm{tg}\,t$

$\lambda^2+4=0$

$\lambda_{1/2}=\pm2i$

$\varphi_1(t)=\cos2t$

$\varphi_2(t)=\sin2t$

$$\Phi(t)=\begin{bmatrix}\cos2t&\sin2t\\ -2\sin2t&2\cos2t\end{bmatrix}$$

$\sin4t=2\sin2t\cos2t$

$$\left.\begin{aligned}\cos2t\cdot c_1'+\sin2t\cdot c_2'&=0\\ -2\sin2t\cdot c_1'+2\cos2t\cdot c_2'&=2\,\mathrm{tg}\,t\end{aligned}\right\}\begin{aligned}&/\cdot2\sin2t\\ &/\cdot\cos2t\end{aligned}$$

$$\left.\begin{aligned}\sin4t\cdot c_1'+2\sin^22t\,c_2'&=0\\ -\sin4t\cdot c_1'+2\cos^22t\,c_2'&=2\,\mathrm{tg}\,t\cdot\cos2t\end{aligned}\right\}+$$

$$2c_2'(\sin^22t+\cos^22t)=2\,\mathrm{tg}\,t\cdot\cos2t$$

$$c_2'=\mathrm{tg}\,t\cdot\cos2t$$

$$c_2=\int\frac{\sin t}{\cos t}\cdot(\cos^2t-\sin^2t)dt=\int\sin t\cdot\cos t\,dt-\int\frac{\sin^3t}{\cos t}dt=\int u\,du-\int\frac{(1-u^2)}{u}(-du)=\frac{\sin^2t}{2}+\ln|\cos t|-\frac{\cos^2t}{2}+c=$$

$u=\sin t$
$du=\cos t\,dt$

$\cos t=u$
$du=-\sin t\,dt$

$$=-\frac{\cos2t}{2}+\ln|\cos t|+c$$

$$c_1' = -c_2' \cdot \frac{\sin 2t}{\cos 2t} = -\operatorname{tg} t \cdot \sin 2t = -2\sin^2 t$$

$$c_1 = \int -2\sin^2 t = -2\left(\frac{t}{2} - \frac{1}{4}\sin 2t\right) + C = -t + \frac{\sin 2t}{2} + C$$

$$\text{OP}: \; x(t) = \underbrace{c_1\cos 2t + c_2 \sin 2t}_{x_H(t)} + \left(-t + \cancel{\frac{\sin 2t}{2}}\right)\cdot \cos 2t + \left(-\cancel{\frac{\cos 2t}{2}} + \ln|\cos t|\right)\cdot \sin 2t =$$

$$= \underbrace{c_1\cos 2t + c_2 \sin 2t}_{x_H(t)} \underbrace{- t\cos 2t + \ln|\cos t| \cdot \sin 2t}_{x_p(t)}$$

Системи ↔ Јне вишег реда

(3) $p, q \in \mathbb{R}\setminus\{0\}$

$$\left.\begin{aligned} x' &= px - qy \\ y' &= qx + py \end{aligned}\right\} (*)$$

2 јне 1. реда ⟶ 1 јна 2. реда

$$qy = px - x'$$

$$y = \frac{px - x'}{q}$$

$$y' = \frac{px' - x''}{q}$$

$$\frac{px' - x''}{q} = q\cdot x + p\cdot\frac{px - x'}{q} \;/\cdot q$$

$$px' - x'' = q^2 x + p^2 x - px' \;\Rightarrow\; x'' - 2px' + (p^2 + q^2)x = 0 \;(\#)$$

домаћи: решити (*) и (#) и упоредити решења

(4) $x''' - 2x'' + x = 0$

1 јна 3. реда ⟶ 3 јне 1. реда

x

$x_1 = x'$

$x_2 = x_1' = x'' \;\Rightarrow\; x_2' = x'''$

(x, x_1, x_2) ⟶ нове пром.

$$x_2' - 2x_2 + x = 0$$

$$\left.\begin{aligned} x_1' &= x_2 \\ x' &= x_1 \\ x_2' &= 2x_2 - x \end{aligned}\right\}$$ (ово није јединствено)

(5) $$\left.\begin{aligned} x''' &= t y' - \sin t \, x' + t \\ y'' &= x'' - \cos(x' \cdot y) \end{aligned}\right\}$$

3. реда, 2. реда } 5 јна 1. реда

$$\left.\begin{array}{l} x \\ x_1 = x' \\ x_2 = x'' \\ y \\ y_1 = y' \end{array}\right\}$$ нове променљиве

$$\left.\begin{array}{l} x''' = x_2' = t \cdot y_1 - \sin t \cdot x_1 + t \\ x' = x_1 \\ x'' = x_1' = x_2 \\ y'' = y_1' = x_2 - \cos(x_1 \cdot y) \\ y' = y_1 \end{array}\right\}$$

лево → изводи
десно → пром.
} систем у нормалном облику

⑥ Решити систем:

$$\left.\begin{array}{l} x'' = x + y \\ y'' - 7y' + 18x'' - 24x' + 6x = 0 \end{array}\right\}$$

2 јне 2. реда → 1 јна 4. реда

$$y = x'' - x \Rightarrow y' = x''' - x' \Rightarrow y'' = x^{(4)} - x''$$

$$x^{(4)} - x'' - 7x''' + 7x' + 18x'' - 24x' + 6x = 0$$

$$x^{(4)} - 7x''' + 17x'' - 17x' + 6x = 0$$

$\vdots$

$x = \ldots$

$y = x'' - x = \ldots$