ЛДЈВРКК

$$F(t, x, x', \dots, x^{(n)}) = 0 \quad , \quad x^{(n)} = \tilde{F}(t, x, x', \dots, x^{(n-1)})$$

- јна линеарна ако је F линеарно по $x, \dots, x^{(n)}$ $\rightsquigarrow$ $a_0(t)\, x + a_1(t) x' + \dots + a_n(t) x^{(n)} = f(t)$
- $f \equiv 0 \rightsquigarrow$ хомогена
- $a_0(t), \dots, a_n(t)$ константне $\rightsquigarrow$ КК
  (реалне)
- ОР : $x(t) = x_H(t) + x_P(t)$

  x_H – опште решење хомогене
  x_P – партикуларно нехомогене

- у случају ЛДЈВРКК : $a_n x^{(n)} + a_{n-1} x^{(n-1)} + \dots + a_1 x' + a_0 x = 0$ (хомогена)

  придружимо (λ-променљива)

  $a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0 = 0$ → карактеристична једначина

  ↳ има n решења у $\mathbb{C}$

  → реална нула p вишеструкости m :

  $e^{pt}, t \cdot e^{pt}, \dots, t^{m-1} e^{pt}$ (m решења)

  → комплексна нула $\alpha \pm i\beta$ вишеструкости m : (2m решења)

  $e^{\alpha t} \cos\beta t, e^{\alpha t} \sin\beta t, \; t e^{\alpha t} \cos\beta t, \; t e^{\alpha t} \sin\beta t, \dots$

  $t^{m-1} e^{\alpha t} \cos\beta t, \; t^{m-1} e^{\alpha t} \sin\beta t$

  →укупно n решења

  ($x_1(t), x_2(t), \dots, x_n(t)$ су лин. независни)

  Т : ОР хомогене је $x(t) = c_1 x_1(t) + \dots + c_n x_n(t)$, $c_i \in \mathbb{R}$

1. а) $x''' - 13x' - 12x = 0 \longrightarrow \lambda^3 - 13\lambda - 12 = 0 \longrightarrow (\lambda+1)\underbrace{(\,\dots\,)}_{\lambda^2 - \lambda - 12} = 0 \longrightarrow \boxed{\begin{aligned}\lambda_1 &= -1\\ \lambda_2 &= -3\\ \lambda_3 &= 4\end{aligned}}$

   $\lambda = 1$ ×
   $\lambda = -1$ ✓

   $\frac{1 \pm \sqrt{1+4 \cdot 12}}{2} = \frac{1 \pm 7}{2}$

   ОР : $x(t) = c_1 e^{-t} + c_2 e^{-3t} + c_3 e^{4t}$, $c_1, c_2, c_3 \in \mathbb{R}$

   б) $x''' - 7x'' + 16x' - 12x = 0$

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$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$

$\lambda = 1$ ×
$\lambda = -1$ ×
$\lambda = 2$ ✓

$(\lambda - 2)\underbrace{(\lambda^2 - 5\lambda + 6)}_{(\lambda-3)(\lambda-2)} = 0 \rightarrow \lambda_1 = \lambda_2 = 2 \rightarrow e^{2t},\ t\cdot e^{2t}$

$\lambda_3 = 3 \rightarrow e^{3t}$

ОР: $x(t) = c_1 e^{2t} + c_2 t e^{2t} + c_3 e^{3t}, \quad c_i \in \mathbb{R}$

в) $x''' - 3x'' + 9x' + 13x = 0$

$\lambda^3 - 3\lambda^2 + 9\lambda + 13 = 0$

$\lambda = 1$ ×
$\lambda = -1$ ✓

$(\lambda + 1)(\lambda^2 - 4\lambda + 13) = 0$

$\downarrow$

$D = 16 - 4\cdot 13 = -36 < 0$ $(\mathbb{C})$

$\lambda_1 = -1 \longrightarrow e^{-t}$

$\lambda_{2/3} = \frac{4 \pm 6i}{2} = 2 \pm 3i \rightarrow e^{2t}\cos 3t,\ e^{2t}\sin 3t$

ОР: $x(t) = c_1 e^{-t} + c_2 e^{2t}\cos 3t + c_3 e^{2t}\sin 3t, \quad c_i \in \mathbb{R}$

г) $x^{(6)} - 4x^{(5)} + 8x^{(4)} - 8x''' + 4x'' = 0$

$\lambda^6 - 4\lambda^5 + 8\lambda^4 - 8\lambda^3 + 4\lambda^2 = 0$

$\lambda^2(\lambda^4 - 4\lambda^3 + 8\lambda^2 - 8\lambda + 4) = 0$

$\lambda = 1$ ×
$\lambda = -1$ ×
$\vdots$

$(\lambda^2 + a\lambda + b)(\lambda^2 + c\lambda + d) = (\lambda^2 - 2\lambda + 2)(\lambda^2 - 2\lambda + 2) = (\lambda^2 - 2\lambda + 2)^2$

$D = 4 - 8 = -4$

$1 \pm i$

$\lambda^3:\ a + c = -4$ ✓
$\lambda^2:\ b + d + ac = 8$
$\lambda:\ ad + bc = -8$
$1:\ bd = 4$

пробамо: $b = d = 2$

$a + c = -4$

$ac = 4$

$\Downarrow$

$a = c = -2$

$\lambda_1 = \lambda_2 = 0 \rightarrow \overset{=1}{e^{0t}},\ t\cdot e^{0t} \rightarrow 1, t$

$\lambda_{3/4} = \lambda_{5/6} = 1 \pm i \rightarrow e^t\cos t,\ e^t\sin t,\ t\cdot e^t\cos t,\ t\cdot e^t\sin t$

$(\alpha = \beta = 1)$

ОР: $x(t) = c_1 + c_2 t + c_3 e^t\cos t + c_4 e^t\sin t + c_5 t e^t\cos t + c_6 t e^t\sin t, \quad c_i \in \mathbb{R}$

(2) Решити Кошијев проблем $x''' + x'' = 0$ (3. реда), $x(0) = 1,\ x'(0) = 0,\ x''(0) = 1$

$\lambda^3 + \lambda^2 = 0$

$\lambda^2(\lambda + 1) = 0$

јна реда к
к услова

} увек јединствено

$$\lambda^3 + \lambda^2 = 0$$

$$\lambda^2(\lambda+1) = 0$$

$$\text{ОР}: x(t) = c_1 + c_2 t + c_3 e^{-t}, \quad c_1, c_2, c_3 \in \mathbb{R}$$

јна реда k
k услова
k константи у ОР
} увек јединствено реш!

$$x(0) = \underline{c_1 + c_3 = 1}$$

$$x'(t) = c_2 - c_3 e^{-t}$$

$$x'(0) = \underline{c_2 - c_3 = 0} \quad \Rightarrow \quad \begin{matrix} c_1 = 0 \\ c_2 = c_3 = 1 \end{matrix} \qquad \underline{x_k(t) = t + e^{-t}}$$

$$x''(t) = c_3 e^{-t}$$

$$x''(0) = \underline{c_3 = 1}$$

Нехомогена: $f \neq 0$

У неким случајевима знамо $f(t) = e^{\alpha t} \cdot \left(P_n(t) \cdot \cos\beta t + Q_m(t) \cdot \sin\beta t\right)$, P_n – пол. ст. n, Q_m – пол. ст. m, $\alpha, \beta \in \mathbb{R}$

s - вишеструкост броја $\alpha \pm i\beta$ као решења карактеристичне једначине од хомогене једначине

$$x_p(t) = t^s \cdot e^{\alpha t} \cdot \left(R_k(t) \cdot \cos\beta t + T_k(t) \cdot \sin\beta t\right)$$

R_k, T_k – пол. ст. k
$k = \max\{n, m\}$

(3) a) $x''' - x'' + x' - x = t^2 + t \qquad (1, \pm i)$

$$\lambda^3 - \lambda^2 + \lambda - 1 = 0$$
$$\Updownarrow$$
$$1, \pm i$$

$$\text{ОР}: x(t) = \underbrace{x_H(t)}_{c_1 e^t + c_2 \cos t + c_3 \sin t} + x_p(t), \quad c_i \in \mathbb{R}$$

$$f(t) = t^2 + t$$

$$\alpha = 0$$

$$\beta = 0 \quad (\sin\beta t = 0, \ \cos\beta t = 1)$$

$$P_n(t) = t^2 + t: \quad \left.\begin{matrix} n = 2 \\ (m = 0) \end{matrix}\right\} k = \max\{m, n\} = 2$$

$$s = ? \qquad \alpha + i\beta = 0 + i \cdot 0 = 0 \Rightarrow s = 0 \quad (0 \notin \{1, i, -i\})$$

није решење

$$x_p(t) = R_2(t) = at^2 + bt + c \longrightarrow \begin{matrix} x_p' = 2at + b \\ x_p'' = 2a \\ x_p''' = 0 \end{matrix}$$

$$x_p''' - x_p'' + x_p' - x_p = t^2 + t$$

$$0 - 2a + (2at + b) - (at^2 + bt + c) = t^2 + t \quad , \forall t \in \mathbb{R}$$

$$\left.\begin{aligned} t^2 &: \ -a = 1 \\ t &: \ 2a - b = 1 \\ 1 &: \ -2a + b - c = 0 \end{aligned}\right\} \quad \begin{aligned} a &= -1 \\ b &= -3 \\ c &= -1 \end{aligned} \qquad x_p(t) = -t^2 - 3t - 1$$

б) $x''' - x'' + x' - x = \cos t + 2e^t$ → nije u opš. obliku

$$\overset{\text{lin. jna}}{L}(x) = \underbrace{f_1(t)}_{x_{p1}} + \underbrace{f_2(t)}_{x_{p2}} = f(t)$$

$(1, \pm i) \rightsquigarrow x_H$

$f_1(t) = \cos t$

$f_2(t) = 2e^t$

$$\text{OP}: x(t) = x_H(t) + \underbrace{\boxed{x_{p1}(t) + x_{p2}(t)}}_{x_p(t)}$$

$$L(x_H + x_{p1} + x_{p2}) = \underbrace{L(x_H)}_{0} + \underbrace{L(x_{p1})}_{f_1} + \underbrace{L(x_{p2})}_{f_2} = 0 + f_1 + f_2 = f(t)$$

f_1: $\alpha = 0$

$\beta = 1$

$$\left.\begin{aligned} Q_m &\equiv 0 \Rightarrow m = -\infty \ (m=0) \\ P_n &\equiv 1 \Rightarrow n = 0 \end{aligned}\right\} k = 0$$

$$\left.\begin{aligned} R_0(t) &= c_1 \\ T_0(t) &= c_2 \end{aligned}\right\} \Rightarrow x_{p1}(t) = t \cdot (c_1 \cos t + c_2 \sin t) \ \dots \ c_1 = c_2 = -\tfrac{1}{4}$$

$$(x_{p1}''' - x_{p1}'' + x_{p1}' - x_{p1} = \cos t)$$

$s = ?$ $\quad \alpha \pm i\beta = 0 \pm i1 = \pm i \Rightarrow s = 1$

f_2: $\left.\begin{aligned} \alpha &= 1 \\ \beta &= 0 \end{aligned}\right\} 1 \pm i \cdot 0 = 1 \Rightarrow s = 1$

$\left.\begin{aligned} n &= 0 \\ (m &= 0) \end{aligned}\right\} k = 0$

$R_k(t) = c \quad \Rightarrow x_{p2}(t) = t \cdot e^t \cdot c \quad --- \ (c = ?)$

$(T_k(t) \ ---)$

↳ nije bitno, jer $\beta = 0$

$$x_{p2}' = ce^t(1 + t)$$

$$x_{p2}'' = ce^t(1 + t + 1)$$

$$x_{p2}''' = ce^t(2 + t + 1)$$

$$x_{p2}''' - x_{p2}'' + x_{p2}' - x_{p2} = 2e^t$$

$$ce^t\big((3+t) - (2+t) + (1+t) - t\big) = 2e^t \Rightarrow c = 1 \Rightarrow x_{p2}(t) = te^t$$

$$\text{OP}: x(t) = \underbrace{c_1 e^t + c_2 \cos t + c_3 \sin t}_{x_H(t)} \underbrace{- \tfrac{t}{4}(\cos t + \sin t)}_{x_{p1}(t)} + \underbrace{te^t}_{x_{p2}(t)} \quad , c_1, c_2, c_3 \in \mathbb{R}$$

в) $x'' - x = \sin^2 t$

(± 1)

$$\sin^2 t = \tfrac{1}{2}(1 - \cos 2t) = \tfrac{1}{2} - \tfrac{1}{2}\cos 2t$$

(± 1)

$$\sin^2 t = \frac{1}{2}(1-\cos 2t) = \underbrace{\frac{1}{2}}_{f_1} - \underbrace{\frac{1}{2}\cos 2t}_{f_2}$$

f_1: $\alpha=0$, $\beta=0$, $s=0$, $n=m=k=0$ $\Big\}$ $x_{p_1}(t)=C \Rightarrow C=-\frac{1}{2}$

f_2: $\alpha=0$, $\beta=2$, $s=0$, $n=m=k=0$ $\qquad x_{p_2}(t) = c_1\cos 2t + c_2 \sin 2t$

г) $x''-4x'+5x = (\sin t + 2\cos t)\cdot e^{2t}$

$(2\pm i)$

$\alpha=2$, $\beta=1$ $\Big\}$ $2\pm i \Rightarrow s=1$, $n=m=k=0$ $\Big\}$ $x_p(t) = t\cdot e^{2t}(c_1\cos t + c_2 \sin t)$ ...

д) $x''-2x'+x = \frac{e^t}{t}$ → није у одг. облику!

$(1\times 2) \rightarrow e^t, t\cdot e^t$

$x_p(t) = e^t\cdot g(t) \qquad (g=?)$

$x_p'(t) = e^t(g(t)+g'(t))$

$x_p''(t) = e^t(g(t)+2g'(t)+g''(t))$

$$\cancel{e^t}(g+2g'+g'') - 2\cancel{e^t}(g+g') + \cancel{e^t}g = \frac{\cancel{e^t}}{t}$$

$\int \ln|t|dt = t\cdot\ln|t| - t$

$$g''+\cancel{2g'}+\cancel{g}-\cancel{2g}-\cancel{2g'}+\cancel{g} = \frac{1}{t} \Rightarrow g'' = \frac{1}{t} \Rightarrow g(t) = \int(\ln|t|+c_1)dt = c_2 + c_1 t + t\cdot\ln|t| - t$$

једно: $c_2=0$, $c_1=1$ $\Big\}$ $x_p(t) = e^t\cdot t\cdot\ln|t|$

ОР: $x(t) = c_1e^t + c_2te^t + e^t t\ln|t|, \quad c_1,c_2\in\mathbb{R}$