Понедељак 01.03.2021.

(1) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Наћи $\det(e^{e^A})$.

$$A \in M_3(\mathbb{R}) \Rightarrow e^A \in M_3(\mathbb{R}) \Rightarrow e^{e^A} \in M_3(\mathbb{R})$$

(6)

$$\delta = \det(e^{e^A}) = e^{tr(e^A)}.$$

$e^A = ?$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E} + \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{N}$$

$$EN = NE = N$$
$$\Downarrow (2)$$
$$e^{E+N} = e^E \cdot e^N$$

$$e^E = \sum_{k=0}^{\infty} \frac{E^k}{k!} = \sum_{k=0}^{\infty} \frac{E}{k!} = E \cdot \sum_{k=0}^{\infty} \frac{1}{k!} = eE$$

$$N^2 = \begin{bmatrix} & & 1 \\ & & \\ 1 & & \end{bmatrix} \cdot \begin{bmatrix} & & 1 \\ & & \\ 1 & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$N^3 = N^2 N = \begin{bmatrix} 1 & & \\ & & \\ & & 1 \end{bmatrix} \begin{bmatrix} & & 1 \\ & & \\ 1 & & \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = N$$

$$N^0 = E$$
$$N^{2k} = N^2, \quad k \geq 1$$
$$N^{2k+1} = N, \quad k \geq 0$$

$$N^4 = N^3 N = NN = N^2$$

$$e^N = \sum_{k=0}^{\infty} \frac{N^k}{k!} = E + \sum_{k=0}^{\infty} \frac{N^{2k+1}}{(2k+1)!} + \sum_{k=1}^{\infty} \frac{N^{2k}}{(2k)!} = E + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \underbrace{\sum_{k=0}^{\infty} \frac{1}{(2k+1)!}}_{sh\,1} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \underbrace{\sum_{k=1}^{\infty} \frac{1}{(2k)!}}_{ch\,1 - 1}$$

($N^{2k+1} = N$, $N^{2k} = N^2$)

$$e = \sum \frac{1}{k!} \qquad e + e^{-1} = 2 \cdot \text{парни}$$

$$e^{-1} = \sum \frac{(-1)^k}{k!} \qquad e - e^{-1} = 2 \cdot \text{непарни}$$

$$\sum_{k=1}^{\infty} \frac{1}{(2k)!} = \frac{e + e^{-1}}{2} - 1 = ch\,1 - 1 \qquad \left(1 = \frac{1}{0!}\right)$$

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)!} = \frac{e - e^{-1}}{2} = sh\,1$$

$$e^N = \begin{bmatrix} \operatorname{ch}1 & 0 & \operatorname{sh}1 \\ 0 & 1 & 0 \\ \operatorname{sh}1 & 0 & \operatorname{ch}1 \end{bmatrix}$$

$$e^A = e^N \cdot e^E = eE \cdot e^N = e \cdot e^N = \begin{bmatrix} e\operatorname{ch}1 & 0 & e\operatorname{sh}1 \\ 0 & e & 0 \\ e\operatorname{sh}1 & 0 & e\operatorname{ch}1 \end{bmatrix}$$

$$d = e^{tr(e^A)} = e^{e(\operatorname{ch}1+\operatorname{ch}1+1)} = e^{e(e+e^{-1}+1)} = e^{1+e+e^2}.$$

② Нека је $A=[a_{ij}]_{i,j=1}^{n} \in M_n(\mathbb{R})$, $a_{ij}\geq 0$, $\forall i\neq j$. Нека је $B=e^A=[b_{ij}]_{i,j=1}^{n}\in M_n(\mathbb{R})$. Доказати да је $b_{ij}\geq 0$, $\forall i,j$.

$$A = \begin{bmatrix} & \geq 0 \\ ? & \\ \geq 0 & \end{bmatrix} \rightsquigarrow B = e^A = \begin{bmatrix} \geq 0 \end{bmatrix}$$

идеја: изоловати ? из матрице A!

уочимо: $A = [\geq 0] \rightsquigarrow A^k = [\geq 0] \rightsquigarrow e^A = [\geq 0]$

$A = A_1 + E_1$ → само ствари на дијагонали

(A_1 ↓ само елементи ≥ 0)

пр.

$$A = \begin{bmatrix} -7 & & \\ & 3 & \\ & & -11 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} -11 & & \\ & -11 & \\ & & -11 \end{bmatrix}$$

$$A_1 = A - E_1 = \begin{bmatrix} 4 & & \\ & 14 & \\ \geq 0 & & 0 \end{bmatrix}$$

$$E_1 = -\underbrace{\max_{1\leq i\leq n} |a_{ii}|}_{=M} \cdot E = -ME$$

$$A_1 = A - E_1 = A + ME = \begin{cases} \underbrace{a_{ij}}_{\geq 0}, & i\neq j \\ \underbrace{a_{ii}+M}_{\geq 0}, & 1\leq i\leq n \end{cases} = [\geq 0]$$

$$a_{ii}+M = a_{ii} + \max_{1\leq k\leq n} |a_{kk}| \geq a_{ii} + |a_{ii}| \geq 0$$

$$e^A = e^{A_1+E_1} = e^{A_1}\cdot e^{E_1}$$

$$A_1E_1 = A_1ME = MA_1 = MEA_1 = E_1A_1$$

$$e^{A} = e^{A_1+E_1} = e^{A_1}\cdot e^{E_1}$$

$A_1E_1 = A_1ME = MA_1 = MEA_1 = E_1A_1$

$$e^{E_1} = \sum_{k=0}^{\infty}\frac{(-ME)^k}{k!} = \sum_{k=0}^{\infty}\frac{(-M)^k}{k!}\cdot E = e^{-M}E = \begin{bmatrix} e^{-M} & & \geq 0 \\ & \ddots & \\ & & e^{-M}\end{bmatrix}$$

$$e^{A} = e^{A_1}\cdot e^{E_1} = \begin{bmatrix} \geq 0 \end{bmatrix}$$

ел. ≥ 0

Решавање нсдјкк у општем случају

$$X' = AX$$

$$X = e^{tA}\cdot c, \quad c\in\mathbb{R}^n$$

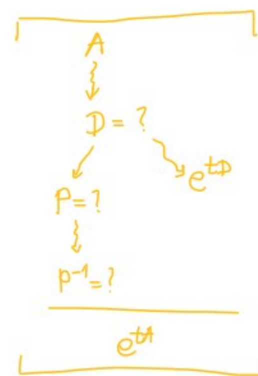
$$A = PDP^{-1} \qquad A\sim D$$

→ D у Жордановој нормалној форми

e^{tD} – знамо да израчунамо

P – матрица прелаза

$$e^{tA} = e^{tPDP^{-1}} \overset{(7)}{=} P\cdot e^{tD}\cdot P^{-1}$$



(4)
$$x_1' = x_1 - x_2 + x_3$$
$$x_2' = x_1 + x_2 - x_3$$
$$x_3' = 2x_1 - x_2$$

$$X' = AX, \quad A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix}, \quad A = PDP^{-1}$$

$$\det(A-\lambda E) = 0$$

$$0 = \begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & 1-\lambda & -1 \\ 2 & -1 & -\lambda \end{vmatrix} = (1-\lambda)^2(-\lambda) + 2 - 1 - 2(1-\lambda) - (1-\lambda) - \lambda = -\lambda(1-\lambda)^2 + 1 - 3(1-\lambda) - 1 =$$

$$= (1-\lambda)(-\lambda(1-\lambda) + 1 - 3) =$$

$$= (1-\lambda)(\lambda^2 - \lambda - 2) =$$

$$= (1-\lambda)(\lambda-2)(\lambda+1)$$

$$\lambda_1 = -1$$

$\lambda_2 = 1$

$\lambda_3 = 2$

реални и различни → 3 собствени вектора

$$D = \begin{bmatrix} -1 & & \\ & 1 & \\ & & 2 \end{bmatrix} \Rightarrow e^{tD} = \begin{bmatrix} e^{-t} & & \\ & e^{t} & \\ & & e^{2t} \end{bmatrix}$$

$\lambda_1 = -1:$ $\quad (A - \lambda_1 E)\gamma_1 = 0$

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$\begin{aligned} 2a - b + c &= 0 \\ a + 2b - c &= 0 \quad \text{(+)} \\ 2a - b + c &= 0 \quad (-1)\ + \end{aligned}$$

$$\begin{aligned} 2a - b + c &= 0 \\ 3a + b \quad &= 0 \\ 0 &= 0 \end{aligned}$$

$b = -3a$

$c = b - 2a = -5a$

$$\begin{bmatrix} a \\ -3a \\ -5a \end{bmatrix} \xrightarrow{a=1} \gamma_1 = \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix}$$

$\lambda_2 = 1:$ $\quad (A - \lambda_2 E)\gamma_2 = 0 \quad \dots \quad \gamma_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_3 = 2:$ $\quad (A - \lambda_3 E)\gamma_3 = 0 \quad \dots \quad \gamma_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} \gamma_1\downarrow & \gamma_2\downarrow & \gamma_3\downarrow \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 1 & 0 \\ -5 & 1 & 1 \end{bmatrix}$$

$\det P = 1 + 0 - 3 + 5 - 0 + 3 = 6$

$$P^{-1} = \frac{1}{\det P} \cdot \operatorname{Adj} P = \frac{1}{6} \cdot \begin{bmatrix} +\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} -3 & 0 \\ -5 & 1 \end{vmatrix} & +\begin{vmatrix} -3 & 1 \\ -5 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ -5 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ -5 & 1 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ -3 & 0 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ -3 & 1 \end{vmatrix} \end{bmatrix}^T = \frac{1}{6} \cdot \begin{bmatrix} 1 & 3 & 2 \\ 0 & 6 & -6 \\ -1 & -3 & 4 \end{bmatrix}^T = \frac{1}{6} \begin{bmatrix} 1 & 0 & -1 \\ 3 & 6 & -3 \\ 2 & -6 & 4 \end{bmatrix}$$

ОР: $X(t) = e^{tA} \cdot c = P e^{tD} P^{-1} c, \quad c \in \mathbb{R}^3$

$P^{-1} c = c_1 \in \mathbb{R}^3$

$X(t) = P \cdot e^{tD} \cdot c_1, \quad c_1 \in \mathbb{R}^3$

(5) $x_1' = -3x_1$

$x_2' = 3x_2 - 2x_3$

$x_3' = x_2 + x_3$

$X' = AX$, $A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 1 \end{bmatrix}$, $\det(A - \lambda E) = 0 \Rightarrow -(\lambda+3)(\lambda^2 - 4\lambda + 5) = 0$

$\lambda_1 = -3$

$\lambda_{2,3} = 2 \pm i$ ← компл. сопр. } разл. собс. вр.

$\lambda_1 = -3$

$D = \begin{bmatrix} -3 & & \\ & 2 & 1 \\ & -1 & 2 \end{bmatrix}$ $\lambda_2 = 2+i$

$\alpha + i\beta \longleftrightarrow \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$

$\lambda_2 = 2+i$, $\alpha = 2$, $\beta = 1$

$(2-i)$

$\lambda_1 = -3$: $(A - \lambda_1 E)v_1 = 0 \ \therefore \ v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_2 = 2+i$: $(2-i)$ $(A - \lambda_2 E)v_2 = 0$ ← компл. собс. век.

$a, b, c \in \mathbb{C}$

$\begin{bmatrix} -3-(2+i) & 0 & 0 \\ 0 & 3-(2+i) & -2 \\ 0 & 1 & 1-(2+i) \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$

$\left.\begin{aligned} (-5-i)a &= 0 \\ (1-i)b - 2c &= 0 \\ b + (-1-i)c &= 0 \end{aligned}\right\}$ (над $\mathbb{C}$)

$(-5-i)a = 0 \Rightarrow a = 0$

$(1-i)b - 2c = 0 \ / \cdot \frac{1+i}{2} \Rightarrow \frac{(1-i)(1+i)}{2} b - (1+i)c = 0 \Rightarrow b - (1+i)c = 0$ X

$(1-i)(1+i) = 1^2 - i^2 = 2$

$b = (1+i)c$

$\begin{bmatrix} 0 \\ (1+i)c \\ c \end{bmatrix}, c \in \mathbb{C}$ пр. $c = 1$, $v_2 = \begin{bmatrix} 0 \\ 1+i \\ 1 \end{bmatrix}$

$\mathrm{Re}\, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(v_3 не треба тражити $\lambda_3 = \overline{\lambda_2} \Rightarrow v_3 = \overline{v_2}$)

$\mathrm{Im}\, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$P = [v_1 \downarrow \ \ \mathrm{Re}(v_2)\downarrow \ \ \mathrm{Im}(v_2)\downarrow] = \begin{bmatrix} 1 & 0 & 0 \\ \ldots \end{bmatrix}$

[0]

$$P = \begin{bmatrix} \nu_1 \downarrow & Re(\nu_2)\downarrow & Im(\nu_2)\downarrow \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\left(P^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -2 \end{bmatrix} \right)$$

$e^{tD} = ?$ D-блоковска матрица

$$e^{tD} = \begin{bmatrix} e^{t(-3)} & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{$e^{t\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}}$}} \\ 0 & & \end{bmatrix} = \begin{bmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{2t}\cos t & e^{2t}\sin t \\ 0 & -e^{2t}\sin t & e^{2t}\cos t \end{bmatrix}$$

(1×1, 2×2)

блок. матр:

$$D = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, \quad D^k = \begin{bmatrix} B_1^k & 0 \\ 0 & B_2^k \end{bmatrix}$$

$$e^{tD} = \begin{bmatrix} e^{tB_1} & \\ & e^{tB_2} \end{bmatrix}$$

прошли час, зад 1г: $\exp\left(t\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}\right) = e^{\alpha t} \cdot R_{\beta t}$

OP: $X(t) = P \cdot e^{tD} \cdot C, \; C \in \mathbb{R}^3$

$C = P^{-1} \cdot c_1$