Понедељак 24.05.2021.

Парцијалне ДЈ 1. реда

(ПДЈ)

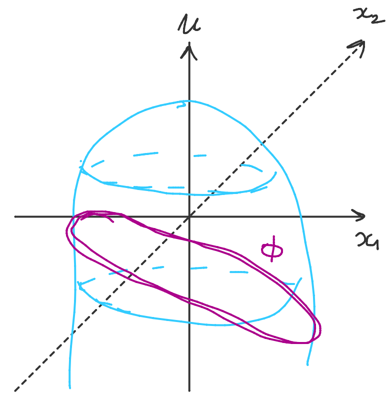
$$u(x_1,\dots,x_n)=? \rightsquigarrow F\left(x_1,\dots,x_n,u,\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n}\right)=0 \quad \text{1. реда}$$

квазилинеарна (КЛ): $\displaystyle\sum_{j=1}^{n} a_j(x_1,\dots,x_n,u)\frac{\partial u}{\partial x_j}=c(x_1,\dots,x_n,u)$

хомогена линеарна (ХЛ): $\displaystyle\sum_{j=1}^{n} a_j(x_1,\dots,x_n)\frac{\partial u}{\partial x_j}=0$ $(c=0)$

Кошијев проблем: наћи решење које садржи задату фигуру Φ.

$$\Phi\subseteq\Gamma(u)$$



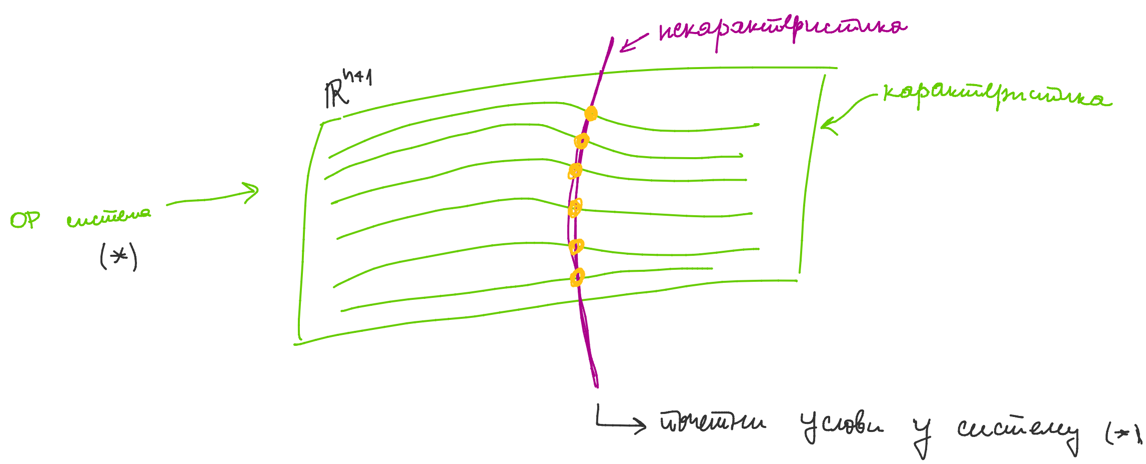
Метода карактеристика (за решавање Кошијевог проблема)

$$(\text{КЛ}) \Rightarrow \quad x_j'(t)=a_j(x_1,\dots,x_n,u)$$

$$u'(t)=c(x_1,\dots,x_n,u)$$

систем карактеристика (*)

$$(\text{ХЛ}) \Rightarrow \quad x_j'(t)=a_j(x_1,\dots,x_n)$$



① Решити Кошијев проблем $u'_x+u'_y+u=1$, $u(x,x+x^2)=\sin x,\ x>0$ $\quad u'_x=\frac{\partial u}{\partial x}$

Кошијев услов

$$(\text{КЛ}) \quad u'_x+u'_y=1-u$$

→ Кошијев услов

(КП) $\qquad u_x' + u_y' = \underbrace{1-u}_{c}$

$$\underbrace{1}_{a_1}\cdot u_x' + \underbrace{1}_{a_2}\cdot u_y' = \underbrace{1-u}_{c}$$

$$\left.\begin{array}{l} x'(t) = 1 \\ y'(t) = 1 \\ u'(t) = 1-u \end{array}\right\} \qquad \begin{array}{l} x(t) = t + c_1 \\ y(t) = t + c_2 \\ u(t) = 1 + c_3 e^{-t} \end{array}$$

$u(x, x+x^2) = \sin x,\ x>0$

C-некарактеристика $\rightsquigarrow \mathcal{H}(s) = (s,\ s+s^2,\ \sin s),\ s>0$
$\subseteq \Gamma(u)$

$$\begin{array}{lll} x(t,s) = t + c_1(s) & x_0(s) = s & x_0(s) = x(0,s) \\ y(t,s) = t + c_2(s) & y_0(s) = s + s^2 & \\ u(t,s) = 1 + c_3(s)\cdot e^{-t} & u_0(s) = \sin s & \end{array}$$

$s = c_1(s)$

$s + s^2 = y_0(s) = c_2(s)$

$\sin s = u_0(s) = 1 + c_3(s)\cdot e^0 \Rightarrow c_3(s) = \sin s - 1$

$$\big(x(t,s),\ y(t,s),\ u(t,s)\big) = \left(t+s,\ t+s+s^2,\ 1 + (\sin s - 1)\cdot e^{-t}\right)$$

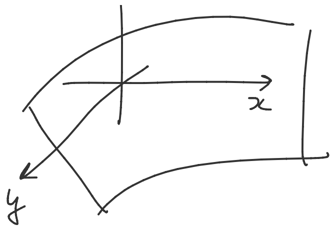
↳ ова твори је решење

$y - x = s^2 \overset{(s>0)}{\Rightarrow} s = \sqrt{y-x}$

$t = x - s = x - \sqrt{y-x}$

$u = 1 + (\sin s - 1)\cdot e^{-t} = 1 + \left(\sin\sqrt{y-x} - 1\right)\cdot e^{-x+\sqrt{y-x}}$

$$u = 1 + (\sin s - 1)\cdot e^{-t} = 1 + \left(\sin\sqrt{y-x} - 1\right)\cdot e^{-x+\sqrt{y-x}}$$



Проверка: $u(s, s+s^2) = \sin s$

$u'_x + u'_y + u = 1$

② $(y+u)\,u'_x + y u'_y = x - y$ $\qquad u\big|_{y=1} = x+1 \longrightarrow u(x,1) = x+1$

$\uparrow$ y

$\underline{(s, 1, s+1) \in \Gamma(u)}$

$x' = y + u$, $\quad x_0(s) = s$

$y' = y$, $\quad y_0(s) = 1$

$u' = x - y$, $\quad u_0(s) = s+1$

$\to$

$y = c_1 e^t$

$x = c_2 e^t + c_3 e^{-t}$

$u = c_2 e^t - c_3 e^{-t} - e^t$

$t = 0$:
$$\left.\begin{aligned} &c_2 + c_3 = s \\ &c_1 = 1 \\ &c_2 - c_3 - 1 = s+1 \end{aligned}\right\} \qquad \begin{aligned} &c_1 = 1 \\ &c_2 = s+1 \\ &c_3 = -1 \end{aligned}$$

решение: $\Big(\underset{x}{\underset{\shortparallel}{(s+1)e^t - e^{-t}}},\ \underset{y}{\underset{\shortparallel}{e^t}},\ \underset{u}{\underset{\shortparallel}{se^t + e^{-t}}}\Big)$

$t = \ln y \Rightarrow e^{-t} = e^{-\ln y} = (y)^{-1} = \frac{1}{y}$

$x = (s+1)y - \frac{1}{y} \Rightarrow s = -1 + \frac{x}{y} + \frac{1}{y^2}$

$$u(x,y) = \left(-1+\frac{x}{y}+\frac{1}{y^2}\right)\cdot y + \frac{1}{y} = x-y+\frac{2}{y}.$$

Метода првих интеграла

идеја: у систему (*) нађи прве интеграле који су независни

(3) $x(y^2-z^2)\cdot\frac{\partial u}{\partial x} - y(x^2+z^2)\frac{\partial u}{\partial y} + z(x^2+y^2)\frac{\partial u}{\partial z} = 0$

$$u\big|_{x=1} = (y+z)^2$$

↳ површ

$u(1,y,z) = (y+z)^2$

$(1,y,z,(y+z)^2) \subseteq \Gamma(u) \subseteq \mathbb{R}^4$

ОР? (ХЛ) $u(x,y,z)$

$$\left.\begin{aligned} x' &= x(y^2-z^2)\\ y' &= -y(x^2+z^2)\\ z' &= z(x^2+y^2)\end{aligned}\right\}$$ 2 прва интеграла ψ_1, ψ_2

$x' = x(y^2-z^2)$

$y' = -y(x^2+z^2)$ ⇝

$z' = z(x^2+y^2)$

$x'_z = \frac{dx}{dz} = \frac{x'}{z'} = \frac{x(y^2-z^2)}{z(x^2+y^2)}$

$y'_z = -\frac{y(x^2+z^2)}{z(x^2+y^2)}$ $\Big\}$ → 2 прва инт.

домаћи: нађи ψ_1 и ψ_2 и показати да су нез.

$\alpha x x' + \beta y y' = \ldots$

$\frac{\alpha x'}{x} + \frac{\beta y'}{y} = \ldots$

$\psi_1 = x^2+y^2+z^2$

$\psi_2 = \frac{yz}{x}$

$\boxed{\text{ОР: } u = \varphi(\psi_1,\psi_2),\ \varphi\in C^1(\mathbb{R}^2)}$

↳ за (ХЛ) једначину

Нађи φ т.д. задовољава Кошијев услов!

$$u(1,y,z) = (y+z)^2$$
$$\| $$
$$\varphi(\underbrace{\psi_1(1,y,z)}_{=\overline{\psi}_1},\ \underbrace{\psi_2(1,y,z)}_{=\overline{\psi}_2})$$

φ? т.д. $\quad \varphi(1+y^2+z^2,\ yz) = (y+z)^2 = y^2+z^2+2yz$

$$\varphi(\overline{\psi}_1, \overline{\psi}_2) = \overline{\psi}_1 - 1 + 2\cdot\overline{\psi}_2$$

Кошијево реш: $\quad u = \varphi(\psi_1, \psi_2) = \psi_1 - 1 + 2\psi_2 = x^2+y^2+z^2-1+\frac{2yz}{x}.$

(4) $\quad x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} + z^2 = 0 \qquad , \ z=1, \ xy = x+y$

$$x' = x^2$$
$$y' = y^2$$
$$z' = -z^2$$

[пробати: експлицитно решити систем, а онда наћи прве интеграле]

$$\frac{dx}{dy} = \frac{x'}{y'} = \frac{x^2}{y^2} \Rightarrow \frac{dx}{x^2} = \frac{dy}{y^2} \ \Big/ \int$$

$$-\frac{1}{x} = -\frac{1}{y} + c_1 \Rightarrow \psi_1 = \frac{1}{x} - \frac{1}{y}$$

$$\frac{dx}{dz} = \frac{x'}{z'} = -\frac{x^2}{z^2} \ \ \text{- - -} \ \ \psi_2 = \frac{1}{x} + \frac{1}{z}$$

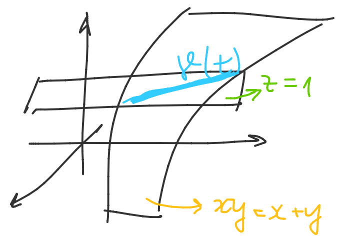
$$\nabla\psi_1 = \left(-\frac{1}{x^2}, \frac{1}{y^2}, 0\right)$$
$$\nabla\psi_2 = \left(-\frac{1}{x^2}, 0, -\frac{1}{z^2}\right)$$

$$\nabla\psi_1 \times \nabla\psi_2 = -\frac{1}{y^2z^2}\vec{i} - \frac{1}{x^2z^2}\vec{j} + \frac{1}{x^2y^2}\vec{k} \neq 0 \quad \Rightarrow \psi_1 \text{ и } \psi_2 \text{ нез!}$$

$\boxed{\text{ОР: } \varphi(\psi_1,\psi_2)=0,\ \varphi\in C^1(\mathbb{R}^2)}$

↳ за (кл) једначину (имплицитно је задато)

КР: Нађи φ тд. $\varphi(\psi_1,\psi_2)=0$ и ово садржи криву $\left.\begin{matrix} z=1 \\ xy=x+y \end{matrix}\right\}$



$\gamma(t) = \left(t, \frac{t}{t-1}, 1\right)$

$\uparrow$ $z=1$

$$xy=x+y$$
$$y(x-1)=x$$
$$y=\frac{x}{x-1}$$

φ? тд: $\varphi\left(\psi_1|_\gamma, \psi_2|_\gamma\right)=0$

$$\varphi(\overline{\psi}_1,\overline{\psi}_2)=0 \Rightarrow \varphi\left(\frac{2}{t}-1, \frac{1}{t}+1\right)=0, \quad \varphi(\overline{\psi}_1,\overline{\psi}_2) = -\overline{\psi}_1+2\overline{\psi}_2-3$$

$$\overline{\psi}_1=\psi_1|_\gamma = \frac{1}{t}-\frac{t-1}{t}=\frac{2}{t}-1$$

$$\overline{\psi}_2=\psi_2|_\gamma = \frac{1}{t}+1$$

$\Rightarrow$ КР: $\varphi(\psi_1,\psi_2)=0$

$$-\psi_1+2\psi_2-3=0$$

$$\frac{1}{x}+\frac{1}{y}+\frac{2}{z}=3 \leftarrow \text{имплицитно задато}$$

Овде може чак експл: $z = \ldots = \dfrac{2}{3-\left(\frac{1}{x}+\frac{1}{y}\right)}$

5) $x(x^2+3y^2)\frac{\partial z}{\partial x}+y(3x^2+y^2)\frac{\partial z}{\partial y}=2z(x^2+y^2)$, $\quad xy=z$, $x^2-y^2=z^2$

$$x' = x(x^2+3y^2)$$

$$y' = y(3x^2+y^2)$$
$$z' = 2z(x^2+y^2)$$

$\rightsquigarrow \psi_1$ и ψ_2 ?

$$\int \frac{x'}{x}\,dt = \int \frac{dx}{x} = \ln|x| + c$$

$$\frac{x'}{x} + \frac{y'}{y} = (x^2+3y^2)+(3x^2+y^2) = 4(x^2+y^2) = 2\frac{z'}{z} \;\Big/ \int dt$$

$$\ln|x| + \ln|y| = 2\ln|z| + c_1$$

$$\vdots$$

$$\psi_1 = \frac{xy}{z^2}$$

$$xx' - yy' = x^2(x^2+3y^2) - y^2(3x^2+y^2) = x^4 - y^4 = (x^2-y^2)(x^2+y^2) = (x^2-y^2)\cdot\frac{z'}{2z}$$

$$\frac{2xx'-2yy'}{x^2-y^2} = \frac{z'}{z} \;\Big/ \int dt \implies \int d(\ln|x^2-y^2|) = \int d(\ln|z|)$$

$$\Downarrow$$

$$\boxed{(\ln|x^2-y^2|)' = \frac{1}{x^2-y^2}\cdot(x^2-y^2)' = \frac{2xx'-2yy'}{x^2-y^2}}$$

$$\ln|x^2-y^2| = \ln|z| + c_2$$

$$\psi_2 = \frac{x^2-y^2}{z}.$$

ψ_1, ψ_2 нез. ← доказати

ОР: $\varphi(\psi_1,\psi_2) = 0,\quad \varphi \in C^1(\mathbb{R}^2)$

КР?
$$\left.\begin{aligned} xy &= z \\ x^2-y^2 &= z^2 \end{aligned}\right\}$$ у пресеку крива c

$$\overline{\psi}_1 = \psi_1\big|_c = \left(\frac{xy}{z^2}\right)\Big|_c \underset{xy=z}{=} \frac{z}{z^2} = \frac{1}{z}$$

записати $\overline{\psi}_1$ и $\overline{\psi}_2$ преко 1 параметра

$$\overline{\psi}_2 = \psi_2\big|_c = \left(\frac{x^2-y^2}{z}\right)\Big|_c = \frac{z^2}{z} = z$$

параметра

$$\overline{\psi_2} = \psi_2\big|_C = \left(\frac{x^2-y^2}{z}\right)\Big|_C = \frac{z^2}{z} = z$$

$$x^2-y^2=z^2$$

$$\Phi\left(\overline{\psi_1}, \overline{\psi_2}\right) = 0$$

$$\overline{\psi_1} = \frac{1}{z}, \quad \overline{\psi_2} = z$$

$$\Phi\left(\overline{\psi_1}, \overline{\psi_2}\right) = \overline{\psi_1}\cdot\overline{\psi_2} - 1$$

Конц. реш.: $\Phi(\psi_1, \psi_2) = 0$

$$\psi_1 \cdot \psi_2 - 1 = 0$$

$$\frac{xy}{z^2}\cdot\frac{x^2-y^2}{z} = 1 \Rightarrow z^3 = xy(x^2-y^2).$$