Среда 30.12.2020.

1) Испитати да ли су токови из зад. (4) јФП.

а) $\phi^t(x) = e^t \cdot x$ (у $\mathbb{R}^n$)

б) $\phi^t(x) = (2+x)e^t - 2t - 2$ (у $\mathbb{R}^1$)

а) $\phi^{t+s}(x) = \phi^t(\phi^s(x))$?

$$\phi^t(\phi^s(x)) = \phi^t(e^s \cdot x) = e^t \cdot e^s \cdot x = e^{t+s} \cdot x = \phi^{t+s}(x) \quad \checkmark$$

б) $$\phi^t(\phi^s(x)) = \phi^t\left((2+x)e^s - 2s - 2\right) = \left(\cancel{2} + (2+x)e^s - 2s - \cancel{2}\right)e^t - 2t - 2 = (2+x)e^{s+t} - 2se^t - 2t - 2$$

$$\phi^{t+s}(x) = (2+x)e^{t+s} - 2t - 2s - 2$$

$\neq$

X

(Т: јФП $\Leftrightarrow$ F аутономно)

а) јесте аутономно

б) није

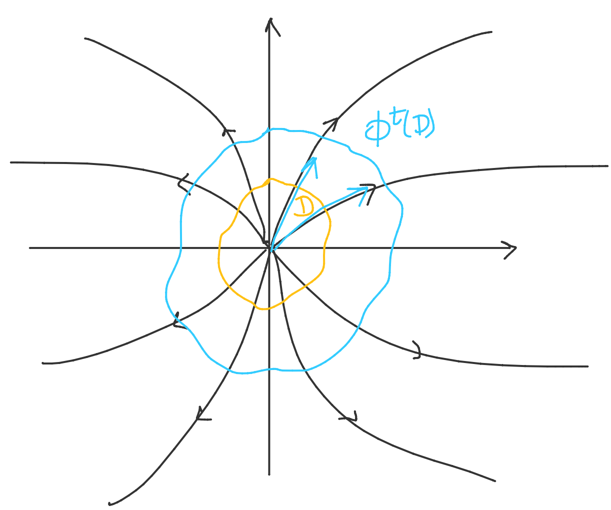
Теорема 103. (Лиувилова теорема - јача верзија.) *Нека је векторско поље F аутономно, ϕ^t решење система (68) и $V(t) := \mathrm{Vol}(\phi^t(D))$, за неки мерљив (компактан) скуп D. Тада је*

$$\frac{dV(t)}{dt} = \int \cdots \int_{\phi^t(D)} \operatorname{div} F \, dy_1 \cdots dy_n, \quad \to \text{промена запр.}$$

где је $\operatorname{div} F = \nabla \cdot F$ *дивергенција векторског поља* F.

$F \rightsquigarrow \phi^t \rightsquigarrow \mathrm{Vol}(\phi^t(D))$

$\operatorname{div} F > 0$ $(=, <)$ $\Rightarrow$ $\dfrac{dV(t)}{dt} > 0$ $(=, <)$ $\Rightarrow$ запремина се повећава (чува, смањује)



стең. $X' = AX$, $F(X) = AX$

$\operatorname{div} F = \operatorname{tr}(A) \rightsquigarrow$ промена запр. зависи од $\operatorname{tr}(A)$

$F = (F_1, \ldots, F_n)$

$$\operatorname{div} F = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \ldots + \frac{\partial F_n}{\partial x_n} = \nabla \cdot F$$

пр. 1) $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, $\lambda_1, \lambda_2 > 0$ (нест. чвор)

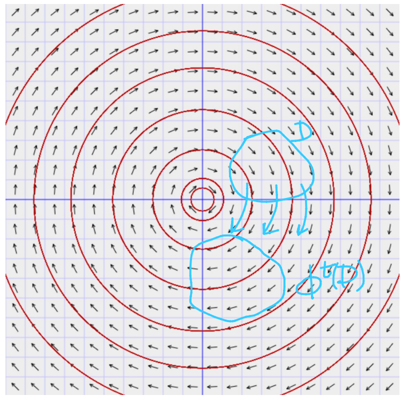
$\operatorname{tr}(A) = \lambda_1 + \lambda_2 > 0 \Rightarrow X' = AX$ повећава запремину

$\operatorname{tr}(A) = \lambda_1 + \lambda_2 > 0 \Rightarrow X' = AX$ повећава запремину

2) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

центар

$\operatorname{tr}(A) = 0$

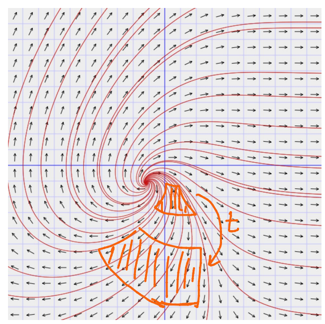


$\phi^t(D) \cong D \longrightarrow$ чува запр.

↑ изометрија, ротација

3) $x_1' = e^{x_1} + x_2$

$x_2' = x_2 - x_1$



$F(x_1, x_2) = (e^{x_1} + x_2, x_2 - x_1)$

$\operatorname{div} F = \frac{\partial}{\partial x_1}(e^{x_1} + x_2) + \frac{\partial}{\partial x_2}(x_2 - x_1)$

$= e^{x_1} + 1 > 0$

$\rightarrow$ повећава запр.

4) $x' = x^2 + 1 \rightsquigarrow \frac{x'}{x^2+1} = 1 \; / \int$

$F(x) = x^2 + 1$

$(F : \mathbb{R} \to \mathbb{R})$

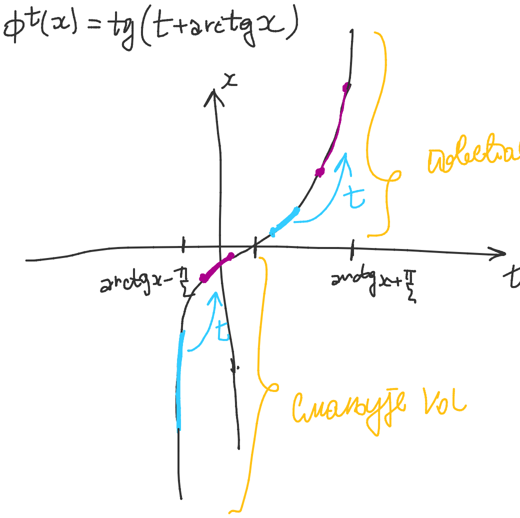
$\operatorname{div} F = \frac{\partial}{\partial x}(x^2 + 1) = 2x$

$\operatorname{arctg} x = t + c$

$x = \operatorname{tg}(t + c)$

$t = 0 \to x_0 \Rightarrow x_0 = \operatorname{tg}(0 + c) \Rightarrow c = \operatorname{arctg} x_0$

$\phi^t(x) = \operatorname{tg}(t + \operatorname{arctg} x)$



повећава Vol

смањује Vol

$\operatorname{div} F = 2x \; \underline{>0} \; \underline{<0}$

Vol у $\mathbb{R}^1 \Leftrightarrow$ дужина дужи

Векторска поља и комутатор (у $\mathbb{R}^n$)

$$F=(F_1,F_2,\dots,F_n) \longleftrightarrow F = F_1\frac{\partial}{\partial x_1}+F_2\frac{\partial}{\partial x_2}+\dots+F_n\frac{\partial}{\partial x_n}$$

$$F: C^\infty(\mathbb{R}^n)\to C^\infty(\mathbb{R}^n)$$

$$F(f) = F_1\frac{\partial f}{\partial x_1}+\dots+F_n\frac{\partial f}{\partial x_n}$$

$$F:\mathbb{R}^n\to\mathbb{R}^n$$

$$F(x_1,\dots,x_n) = (F_1(x_1,\dots,x_n),\dots,F(x_1,\dots,x_n))$$

- $F(\lambda f+\mu g) = \lambda F(f)+\mu F(g)$
- $F(fg) = F(f)g+fF(g)$

комутатор: $\underbrace{[F,G]}_{\text{вект. поље}} = F\circ G - G\circ F$

$$[F,G]=0 \iff F\circ G = G\circ F$$

$$\underset{C^\infty}{[F,G](f)} = \underbrace{F(\underbrace{G(f)}_{C^\infty})}_{C^\infty} - \underbrace{G(\underbrace{F(f)}_{C^\infty})}_{C^\infty}$$

смене променљиве: $\varphi:(x_1,\dots,x_n)\longmapsto(y_1,\dots,y_n)$

$$(\forall k)\quad \frac{\partial}{\partial y_k} = \sum_{l=1}^{n}\frac{\partial x_l}{\partial y_k}\cdot\frac{\partial}{\partial x_l}$$

$\varphi_* F = G \longrightarrow$ у коорд. $y_1,\dots,y_n$

$\wr$ у коорд. $x_1,\dots,x_n$

push forward (гурање)

② $X = -y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}$ у $\mathbb{R}^2$ $\qquad X\longleftrightarrow(-y,x)$

Нађи ток ϕ^t овог в.п.

$$\frac{d\phi^t}{dt} = \underset{C^\infty}{X(\phi^t)}, \quad \phi^0=id$$

X у тачки ϕ^t

$$\phi^t(x) = (u(t),v(t))$$

$$(u'(t), v'(t)) = (-v(t), u(t))$$

$$\left.\begin{array}{l} u' = -v \\ v' = u \end{array}\right\} \dots \qquad \begin{array}{l} u = c_1 \sin t + c_2 \cos t \\ v = -c_1 \cos t + c_2 \sin t \end{array}$$

$$\phi^0(x_0) = x_0 = (x_1, x_2)$$

$$u(0) = c_2 \qquad c_2 = \pi_1 x_0 = x_1$$

$$v(0) = -c_1 \qquad c_1 = -\pi_2 x_0 = -x_2$$

$$\phi^t \underbrace{(x_1, x_2)}_{\in \mathbb{R}^2} = (-x_2 \sin t + x_1 \cos t,\ x_2 \cos t + x_1 \sin t)$$

③ Нађи в.п. за ток $\phi^t(x, y) = (\underbrace{x \cos t + y \sin t}_{f_1}, \underbrace{-x \sin t + y \cos t}_{f_2})$

$$\frac{d\phi^t(x,y)}{dt} = \underset{?}{X}(\phi^t(x,y))$$

$$\frac{d\phi^t(x,y)}{dt} = (-x \sin t + y \cos t,\ -x \cos t - y \sin t) = (f_2, -f_1) = X(\phi^t(x,y)) = X(f_1, f_2)$$

$$\Rightarrow X(x,y) = (y, -x)$$

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

④ $X = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + 2x_3\frac{\partial}{\partial x_3}$, $\quad \phi^t(x_1, x_2, x_3) = (-x_2 \sin t + x_1 \cos t,\ x_2 \cos t + x_1 \sin t,\ e^t x_3)$

а) ψ^t ?

б) Ψ и Φ да ли комутирају? $\quad (\psi^t \circ \phi^1 = \phi^1 \circ \psi^t)$

$$\begin{tikzcd} \bullet \arrow[r, "\phi^1"] \arrow[d, "\psi^t"'] \arrow[dr, dashed] & \bullet \arrow[d, "\psi^t"] \\ \bullet \arrow[r, "\phi^1"'] & \bullet \end{tikzcd}$$

а) $\psi^t = (u, v, w)$

$$\frac{d\psi^t}{dt} = X(\psi^t) \Rightarrow \begin{array}{ll} u' = u & \Rightarrow u(t) = c_1 e^t \\ v' = v & \Rightarrow v(t) = c_2 e^t \\ w' = 2w & \Rightarrow w(t) = c_3 e^{2t} \end{array}$$

$$\left.\psi^t(x_1, x_2, x_3)\right|_{t=0} = (x_1, x_2, x_3) \Big\} \Rightarrow \psi^t(x_1, x_2, x_3) = (x_1 e^t, x_2 e^t, x_3 e^{2t})$$

$$\left.\psi^t(x_1,x_2,x_3)\right|_{t=0} = (x_1,x_2,x_3)$$
$$\parallel$$
$$(c_1,c_2,c_3)$$
$$\Rightarrow \psi^t(x_1,x_2,x_3) = (x_1e^t, x_2e^t, x_3e^{2t})$$

б) $\psi^t(x_1,x_2,x_3) = (x_1e^t, x_2e^t, x_3e^{2t})$

$\phi^s(x_1,x_2,x_3) = (-x_2\sin s + x_1\cos s,\ x_2\cos s + x_1\sin s,\ e^s x_3)$

$$\psi^t\circ\phi^s(x_1,x_2,x_3) = \psi^t(\ ---\ ) = \big((-x_2\sin s + x_1\cos s)e^t,\ (x_2\cos s + x_1\sin s)e^t,\ (e^s x_3)\cdot e^{2t}\big)$$
$$\parallel$$
$$\phi^s\circ\psi^t(x_1,x_2,x_3) = \phi^s(\_\_\_\_) = (-x_2e^t\sin s + x_1e^t\cos s,\ x_2e^t\cos s + x_1e^t\sin s,\ e^s\cdot x_3e^{2t})$$

$\Rightarrow \psi$ и ϕ комутирају

домаћи: Наћи в.п. Y за ϕ^t и израчунати $[X,Y]$.

Т: $F \rightsquigarrow \phi^t$
$G \rightsquigarrow \psi^t$
$[F,G]=0 \iff \phi$ и ψ комутирају

⑤ $X_1 = \frac{\partial}{\partial x}$, $Y_1 = \frac{\partial}{\partial y}$

$X_2 = \frac{\partial}{\partial x}$, $Y_2 = (1+x^2)\frac{\partial}{\partial y}$

Наћи $[X_1,Y_1]$, $[X_2,Y_2]$.

$$[X_1,Y_1](f) = X_1(Y_1(f)) - Y_1(X_1(f)) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}f\right) - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}f\right) = \frac{\partial^2 f}{\partial x\partial y} - \frac{\partial^2 f}{\partial y\partial x} = 0$$

↳ 2. парц. изводи комутирају

$$[X_2,Y_2](f) = \frac{\partial}{\partial x}\left((1+x^2)\frac{\partial f}{\partial y}\right) - (1+x^2)\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial(1+x^2)}{\partial x}\cdot\frac{\partial f}{\partial y} + (1+x^2)\cdot\frac{\partial^2 f}{\partial x\partial y} - (1+x^2)\frac{\partial^2 f}{\partial y\partial x} = 2x\cdot\frac{\partial f}{\partial y}$$

$$\Rightarrow [X_2,Y_2] = 2x\cdot\frac{\partial}{\partial y} = (0, 2x)$$

⑥ $[X, fY] = X(f)Y + f\cdot[X,Y]$, $f\in C^\infty(\mathbb{R}^n)$

$$[X,fY](g) = \underbrace{X(fY(g))}_{\text{производ}} - fY(X(g)) = X(f)\cdot Y(g) + f\cdot X(Y(g)) - f\cdot Y(X(g)) = X(f)\cdot Y(g) + f\cdot(X(Y(g)) - Y(X(g))) =$$
$$= (X(f)\cdot Y + f\cdot[X,Y])(g)$$

$[X, fY](g) = \underbrace{X(fY(g))}_{\text{производ фја } f \text{ и } Y(g)} - fY(X(g)) - X(f) \cdot Y(g) \ldots$

$= (X(f) \cdot Y + f \cdot [X, Y])(g)$

$\Rightarrow [X, fY] = X(f) \cdot Y + f \cdot [X, Y]$

⑦ $\{xy > 0\} \subseteq \mathbb{R}^2$, $X = \frac{x}{y}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$, $Y = 2\sqrt{xy}\,\frac{\partial}{\partial x}$

а) $[X, Y]$

б) $x = uv^2$, $y = u$

изразити X и Y у новим коорд.

$\Big\} \Rightarrow \frac{x}{y} = v^2 \Rightarrow v = \sqrt{\frac{x}{y}}$, $u = y$

а) $[X, Y] = \left(\frac{x}{y}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(2\sqrt{xy}\,\frac{\partial}{\partial x}\right) - 2\sqrt{xy}\,\frac{\partial}{\partial x}\left(\frac{x}{y}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)$

$= \frac{x}{y}\frac{\partial}{\partial x}\left(2\sqrt{xy}\,\frac{\partial}{\partial x}\right) + \frac{\partial}{\partial y}\left(2\sqrt{xy}\,\frac{\partial}{\partial x}\right) - 2\sqrt{xy}\,\frac{\partial}{\partial x}\left(\frac{x}{y}\frac{\partial}{\partial x}\right) - 2\sqrt{xy}\,\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right) =$

$= \frac{x}{y}\cdot\left(\sqrt{\frac{y}{x}}\,\frac{\partial}{\partial x} + 2\sqrt{xy}\,\frac{\partial^2}{\partial x^2}\right) + \left(\sqrt{\frac{x}{y}}\,\frac{\partial}{\partial x} + 2\sqrt{xy}\,\frac{\partial^2}{\partial y \partial x}\right) - 2\sqrt{xy}\left(\frac{1}{y}\frac{\partial}{\partial x} + \frac{x}{y}\frac{\partial^2}{\partial x^2}\right) - 2\sqrt{xy}\,\frac{\partial^2}{\partial x \partial y} =$

$= 2\sqrt{\frac{x}{y}}\,\frac{\partial}{\partial x} - 2\sqrt{\frac{x}{y}}\,\frac{\partial}{\partial x} = 0$

б) $\frac{\partial}{\partial u} = \frac{\partial x}{\partial u}\cdot\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\cdot\frac{\partial}{\partial y} = v^2\frac{\partial}{\partial x} + \frac{\partial}{\partial y} = \frac{x}{y}\frac{\partial}{\partial x} + \frac{\partial}{\partial y} = X$

$\frac{\partial x}{\partial u} = \frac{\partial(uv^2)}{\partial u} = v^2$

$\frac{\partial y}{\partial u} = \frac{\partial u}{\partial u} = 1$

$\frac{\partial}{\partial v} = \frac{\partial x}{\partial v}\,\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\,\frac{\partial}{\partial y} = 2uv\,\frac{\partial}{\partial x} + 0\,\frac{\partial}{\partial y} = 2\sqrt{xy}\,\frac{\partial}{\partial x} = Y$

$\varphi(x, y) = \left(\underset{u}{y}, \underset{v}{\sqrt{\frac{x}{y}}}\right)$, $\varphi_* X = \frac{\partial}{\partial u}$

$\varphi_* Y = \frac{\partial}{\partial v}$

$\varphi_*[X, Y] \overset{\text{пред.}}{=} [\varphi_* X, \varphi_* Y] = \left[\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right] = 0$

$\Rightarrow [X, Y] = 0$

⑧ Да ли постоји смена, т.д. X_2, Y_2 из зад. ⑤ буду координатна?

$X_2 = \frac{\partial}{\partial x}$

$\left.\begin{matrix} u(x, y) \\ v(x, y) \end{matrix}\right\}?$

и $\ldots \frac{\partial}{\ldots}$

$X_2 = \frac{\partial}{\partial x}$

$Y_2 = (1+x^2)\frac{\partial}{\partial y}$

$\left.\begin{matrix} u(x,y) \\ v(x,y) \end{matrix}\right\}$?

$$\frac{\partial}{\partial x} = X_2 = \frac{\partial}{\partial u} = \frac{\partial x}{\partial u}\cdot\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\cdot\frac{\partial}{\partial y}$$

$$(1+x^2)\frac{\partial}{\partial y} = Y_2 = \frac{\partial}{\partial v} = \frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y}$$

$$\left\}\begin{matrix} \frac{\partial x}{\partial u} = 1, \quad \frac{\partial y}{\partial u} = 0 \\ \frac{\partial x}{\partial v} = 0, \quad \frac{\partial y}{\partial v} = (1+x^2) \end{matrix}\right.$$

... тешко

$$\left.\begin{matrix} \varphi_* X_2 = \frac{\partial}{\partial u} \\ \varphi_* Y_2 = \frac{\partial}{\partial v} \end{matrix}\right\} \Rightarrow \varphi_*[X_2, Y_2] = 0 \overset{(\varphi_*)^{-1}}{\Longrightarrow} [X_2, Y_2] = 0 \quad ϟ$$

у (u,v) коорд. у (x,y) коорд

(9) $X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$

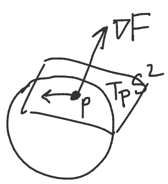
$Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$ у $\mathbb{R}^3$ сфера $S^2 = \{x^2+y^2+z^2=1\} \subseteq \mathbb{R}^3$

$Z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$

а) X, Y, Z тангентни на сферу

б) $[X,Y], [X,Z], [Y,Z]$ тангентни на сферу

а)



$x^2+y^2+z^2=1$

$F(x,y,z) = c \Rightarrow (\nabla F)_p \perp T_pS^2$

X тангентна на $S^2 \Leftrightarrow X \perp \nabla F$

$\nabla F = (2x, 2y, 2z)$

$\langle \nabla F, X\rangle = \langle (2x,2y,2z), (0,-z,y)\rangle = 0$ ✓

Y, Z – слично

б) $[X,Y] = \dots = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$

$\langle \nabla F, [X,Y]\rangle = 0$