Понедељак 19.10.2020.

Типови ДЈ

1 $x' = f(\alpha t + \beta x + \gamma)$, $f \in C(a,b)$, $\alpha, \beta \in \mathbb{R}\setminus\{0\}$, $\gamma \in \mathbb{R}$

смена: $x(t) \rightsquigarrow y(t)$

$$\left.\begin{aligned} & y(t) = \alpha t + \beta x(t) + \gamma \\ & y' = \alpha + \beta x' \Rightarrow x' = \frac{y' - \alpha}{\beta} \end{aligned}\right\} \rightsquigarrow \text{РП}$$

(1) а) $x' = x + 2t - 3$

б) $x' = (x+t)^2$ $\qquad [f(\cdot) = \cdot^2]$

б) $y = x + t$

$y' = x' + 1 \Rightarrow x' = y' - 1$

$$y' - 1 = y^2 \Rightarrow y' = y^2 + 1 \Rightarrow \frac{dy}{y^2+1} = dt \ \Big/ \int$$

$$\operatorname{tg}\Big| \ \operatorname{arctg} y = t + c, \quad c \in \mathbb{R}$$

$$y = \operatorname{tg}(t+c)$$

$$x + t = \operatorname{tg}(t+c)$$

$$\boxed{x = \operatorname{tg}(t+c) - t, \quad c \in \mathbb{R}}$$

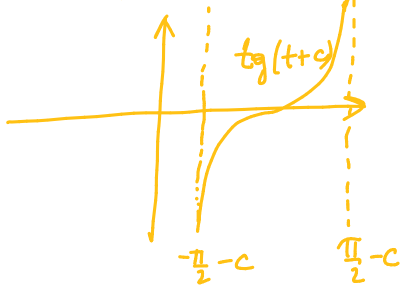
НАП: дефинисаност решења

$$t + c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$-\frac{\pi}{2} - c < t < \frac{\pi}{2} - c$$

∞ много решења, свако решење је деф. на различитом интервалу

+ решења НЕ могу да се продуже, јер $\to +\infty$



Т о продужењу решења (касније на предавањима)

2 $x' = f\left(\frac{x}{t}\right)$, $f \in C(a,b)$ $\qquad$ (хомогена)

смена: $y(t) = \frac{x(t)}{t}$ $\quad (y' = \ldots)$

$$\left.\Rightarrow x = yt \ \Big/ ' \Rightarrow x' = y' \cdot t + y \cdot 1 = y + ty'\right\} \rightsquigarrow \text{РП}$$

$$\Rightarrow x = yt \,/'\ \Rightarrow x' = y'\cdot t + y\cdot 1 = y + ty'$$

(2) а) $x' = e^{\frac{x}{t}} + \frac{x}{t}$

б) $x' = -\frac{x^2+t^2}{2xt}$

в) $x' = \frac{x^2-2xt-t^2}{x^2+2xt-t^2}$

г) $t\sin\frac{x}{t}\cdot x' = x\cdot\sin\frac{x}{t} + t$

в) $$x' = \frac{x^2-2xt-t^2}{x^2+2xt-t^2} \overset{:t^2}{=} \frac{\left(\frac{x}{t}\right)^2 - 2\frac{x}{t} - 1}{\left(\frac{x}{t}\right)^2 + 2\frac{x}{t} - 1}$$

$0\ \not$

$$\left[f(\cdot) = \frac{\cdot^2 - 2\cdot - 1}{\cdot^2 + 2\cdot - 1} \right] \quad ; \quad y = \frac{x}{t}$$

$$x = yt$$
$$x' = y't + y$$

$$y't + y = x' = \frac{y^2-2y-1}{y^2+2y-1}$$

$$y't = \frac{y^2-2y-1}{y^2+2y-1} - y = \frac{y^2-2y-1-y(y^2+2y-1)}{y^2+2y-1} = \frac{-y^3-y^2-y-1}{y^2+2y-1}$$

генимо са 0?

$$\frac{y^2+2y-1}{y^3+y^2+y+1}\,dy = -\frac{dt}{t} \ \Big/ \int$$

$$-\ln|t| + c = \int \frac{y^2+2y-1}{y^3+y^2+y+1}\,dy$$

$$\frac{y^2+2y-1}{y^3+y^2+y+1} = \frac{A}{y+1} + \frac{By+C}{y^2+1} = \frac{A(y^2+1)+(By+C)(y+1)}{(y+1)(y^2+1)} = \frac{y^2(A+B)+y(B+C)+(A+C)}{(y+1)(y^2+1)}$$

$$y^3+y^2+y+1 = (y+1)(y^2+1)$$

$$\left.\begin{array}{l} A+B=1 \\ B+C=2 \\ A+C=-1 \end{array}\right\} + \quad 2(A+B+C) = 2 \Rightarrow A+B+C=1$$

$$A=-1,\ B=2,\ C=0$$

$$e^{\llcorner} \backslash \ -\ln|t| + c = \int \cdots = \int \frac{-1}{y+1}\,dy + \int \frac{2y\,dy}{y^2+1} = -\ln|y+1| + \ln|y^2+1| = \ln\left|\frac{y^2+1}{y+1}\right|$$

$$\frac{e^c}{|t|} = \left|\frac{y^2+1}{y+1}\right| \Rightarrow \frac{y^2+1}{y+1} = \frac{c_1}{t}, \quad c_1 \in \mathbb{R}\setminus\{0\} \ (\pm e^c)$$

$$y = \frac{x}{t} \qquad \frac{\left(\frac{x}{t}\right)^2+1}{\ } \overset{:t^2}{\ } = \frac{c_1}{\ }$$

$y=\frac{x}{t}$

$$\frac{\left(\frac{x}{t}\right)^2+1}{\left(\frac{x}{t}\right)+1}=\frac{c_1}{t} \quad \cdot t^2$$

$$\boxed{\frac{x^2+t^2}{x+t}=c_1 \ , c_1\in\mathbb{R}\setminus\{0\} \quad (1)}$$

интегрално зададено решение

експлицитно зададено $x(t)=\ldots$

$y^3+y^2+y+1=0$?

$(y+1)\underbrace{(y^2+1)}_{\neq 0}=0 \to y=-1$?

$\frac{x}{t}=-1 \Rightarrow x=-t$ решение? (2)

$x+t=0$

$$-1=x'=\frac{t^2+2t^2-t^2}{t^2-2t^2-t^2}=\frac{2t^2}{-2t^2}=-1 \checkmark$$

ОР: (1)+(2)

ГНАП: точките - делими са $x^2+2xt-t^2 \overset{?}{=} 0$ $\Big\}$ $x'=\frac{---}{0}\to\infty \to$ вертикален наклон „$x'=\infty$"

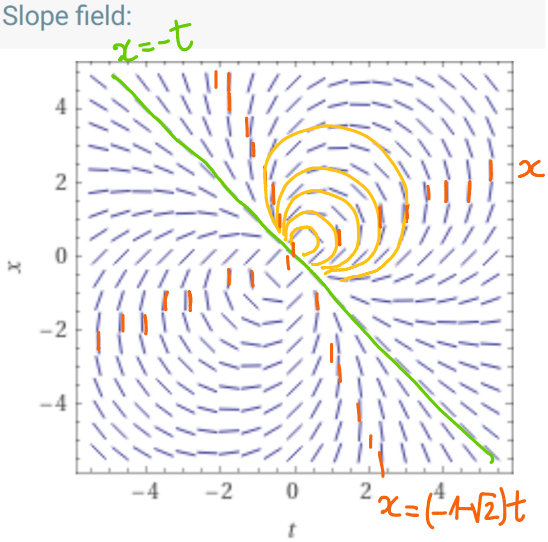
$y^2+2y-1=0$

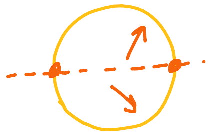
$y=-1\pm\sqrt{2} \Rightarrow x=(-1\pm\sqrt{2})t$

$x=(-1-\sqrt{2})t$

$x=(-1+\sqrt{2})t$

Slope field:





https://tinyurl.com/linkKaWA1

3) $x'=f\left(\frac{a_1t+b_1x+c_1}{a_2t+b_2x+c_2}\right)$, $f\in C(a,b)$

↓

РП v ХОМ

1° $c_1=c_2=0$: $x'=f\left(\frac{a_1t+b_1x+c_1}{a_2t+b_2x+c_2}\right)=f\left(\frac{a_1t+b_1x}{a_2t+b_2x}\right)=f\left(\frac{a_1+b_1\frac{x}{t}}{a_2+b_2\frac{x}{t}}\right)=g\left(\frac{x}{t}\right)\to$ ХОМ

2° $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=a_1b_2-a_2b_1\neq 0$

2° $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \neq 0$

$\alpha, \beta \in \mathbb{R}$

$x = v + \alpha$

$t = u + \beta$ $\quad \frac{dt}{du} = 1$

$v = x - \alpha$

$\frac{dv}{dx} = 1$

$x(t) \rightsquigarrow v(u) = x(u) - \alpha = x(t-\beta) - \alpha$

$a_1t + b_1x + c_1 = a_1(u+\beta) + b_1(v+\alpha) + c_1 = \boxed{a_1u + b_1v} + \overbrace{(a_1\beta + b_1\alpha + c_1)}^{=0}$

$a_2t + b_2x + c_2 = a_2(u+\beta) + b_2(v+\alpha) + c_2 = \boxed{a_2u + b_2v} + \underbrace{(a_2\beta + b_2\alpha + c_2)}_{=0}$

хом.

α, β бирамо да буду решења система $\left.\begin{aligned} a_1\beta + b_1\alpha + c_1 &= 0 \\ a_2\beta + b_2\alpha + c_2 &= 0 \end{aligned}\right\}$

$\det \neq 0 \Rightarrow \exists_1 \alpha, \beta$

$x' \rightsquigarrow ?$ $\quad x' = \frac{dx}{dt}$

$v' = \frac{dv}{du} = \underbrace{\boxed{\frac{dv}{dx}}}_{=1} \cdot \frac{dx}{dt} \cdot \underbrace{\boxed{\frac{dt}{du}}}_{=1} = \frac{dx}{dt} = x'$

3° $\det = 0$

$a_1b_2 - a_2b_1 = 0$

$a_1b_2 = a_2b_1$

3.1° ако је неки нула $\Rightarrow$ тривијално.

3.2° $\frac{a_1}{a_2} = \frac{b_1}{b_2} = k \in \mathbb{R}$ $\quad : \quad \frac{a_1t + b_1x + c_1}{a_2t + b_2x + c_2} = \frac{a_2kt + b_2kx + c_1}{a_2t + b_2x + c_2} = \frac{k\boxed{(a_2t + b_2x)} + c_1}{\boxed{a_2t + b_2x} + c_2}$

своди се на $\boxed{1}$

$\Rightarrow a_1 = ka_2$

$b_1 = kb_2$

(3) a) $(x + 2t - 2)x' = x - t - 1$

б) $x' = \frac{t + x + 4}{t + x - 6}$

a) $x' = \frac{x - t - 1}{x + 2t - 2}$ $\quad =0?$

$x + 2t - 2 = 0$

$x = -2t + 2$?

$(x + 2t - 2)x' = x - t - 1$

$0 = 0 \cdot x' = -3t + 1$ ✗

$\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = 1 \cdot 2 - (-1) \cdot 1 = 3 \neq 0$

$x = v + \alpha$

$t = u + \beta$

$x - t - 1 = (v + \alpha) - (u + \beta) - 1 = v - u + \overbrace{(\alpha - \beta - 1)}^{=0}$

$x + 2t - 2 = (v + \alpha) + 2(u + \beta) - 2 = v + 2u + \underbrace{(\alpha + 2\beta - 2)}_{=0}$ $\}$ $-$

$-3\beta + 1 = 0$

$v' = \frac{dv}{du} = \frac{dx}{dt} = x'$

$x+2t-2=$

$-3\beta+1=0$

$\beta=\frac{1}{3} \to \alpha=\frac{4}{3}$

$$x'=v'=\frac{v-u}{v+2u}=\frac{\frac{v}{u}-1}{\frac{v}{u}+2} \quad (\text{хом})$$

$$w=\frac{v}{u} \to v=w\cdot u /' \to v'=w'u+w$$

$$w'u+w=\frac{w-1}{w+2} \Rightarrow \dots \quad \frac{w+2}{w^2+w+1}\,dw=-\frac{du}{u} \Big/ \int$$

$=0$? $w^2+w+1>0$ X

$$w(u) \rightsquigarrow v(u)=u\cdot w(u)$$

$x(t)$

б) $x'=\dfrac{t+x+4}{t+x-6}$

$$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}=1-1=0$$

$=0$? $t+x-6\neq 0$ због тешавке!

директно [1], а не [3]: $t+x=y(t)$

$\therefore y'=x'+1$

$y'-1=\dfrac{y+4}{y-6}$ ...

[4] Линеарна ДЈ 1. реда

$p,q:(a,b)\to\mathbb{R}$ непр.

$$x'+p(t)x=q(t)$$

линеарно

извод је увек линеаран

$g(t)\equiv 0 \to$ хомогена ЛДЈ 1.р.

$g(t)\neq 0 \to$ нехомогена ЛДЈ 1.р.

предавања $\to$

$$\boxed{x(t)=e^{-\int p\,dt}\cdot\left(C+\int e^{\int p\,dt}\cdot q\,dt\right), \quad C\in\mathbb{R}}$$

$\int p\,dt=\int_{t_0}^{t} p(u)\,du$

↳ прим. ф-ја је иста на оба места

(4) а) $tx'-x=t^3$

б) $x'+x=\dfrac{1}{1+e^{2t}}$

в) $x'-2tx=6te^{t^2}$

г) $tx'+ax+t^n=0, \quad a\in\mathbb{R}, n\in\mathbb{N}$

в) $p(t)=-2t$

$q(t)=6te^{t^2}$

$$\int p\,dt=\int -2t\,dt=-t^2 \ (+C)$$

$$\int e^{\int p\,dt} \cdot q\,dt = \int e^{-t^2} \cdot 6t \cdot e^{t^2}\,dt = \int 6t\,dt = 3t^2 \; (+c)$$

$$x(t) = e^{-(-t^2)} \cdot (c + 3t^2) = e^{t^2}(c + 3t^2) \quad , c \in \mathbb{R}$$

$$= \underbrace{c \cdot e^{t^2}}_{\text{решење хомогене}} + \underbrace{e^{t^2} \cdot 3t^2}_{\text{неко партикуларно решење}}$$

а) $tx' - x = t^3$

$$\boxed{\begin{matrix} p(t) = -1 \\ q(t) = t^3 \end{matrix}} \; \cancel{X}$$

$$x' - \frac{1}{t}x = t^2 \rightsquigarrow \begin{matrix} p(t) = -\frac{1}{t} \\ q(t) = t^2 \end{matrix}$$

5) Нађи реш ДЈ. $x' \sin t - x \cos t = -\frac{\sin^2 t}{t^2}$ уз. $\lim\limits_{x \to \infty} x(t) = 0$.

I начин: линеарна $\quad x' - \frac{\cos t}{\sin t}x = -\frac{\sin t}{t^2} \; \ldots$

II начин: $\quad x' \sin t - x \cos t$

$$(x \sin t)' = x' \sin t + x \cdot \cos t$$

$$\left(\frac{x}{\sin t}\right)' = \frac{x' \sin t - x \cos t}{\sin^2 t}$$

$$\frac{x' \sin t - x \cos t}{(\sin t)^2} = -\frac{1}{t^2}$$

$$\left(\frac{x}{\sin t}\right)' = -\frac{1}{t^2} \quad / \int$$

$$\frac{x}{\sin t} = \frac{1}{t} + c \quad , c \in \mathbb{R}$$

$$\boxed{\text{ОР: } x = \frac{\sin t}{t} + c \cdot \sin t \; , c \in \mathbb{R}}$$

$\lim\limits_{x \to \infty} x(t) = 0$: $\quad \frac{\sin t}{t} \xrightarrow[t \to \infty]{} 0$

$$c \cdot \sin t \xrightarrow[t \to \infty]{} 0 \quad \Rightarrow \boxed{c = 0}$$

$$\Rightarrow \boxed{\;\ldots\; \frac{\sin t}{\ldots}}$$

$$c \cdot \sin t \xrightarrow[t \to \infty]{} 0 \implies \boxed{c = 0}$$

ПР: $\boxed{x(t) = \dfrac{\sin t}{t}}$