Понедељак 16.11.2020.

Фазни портрети

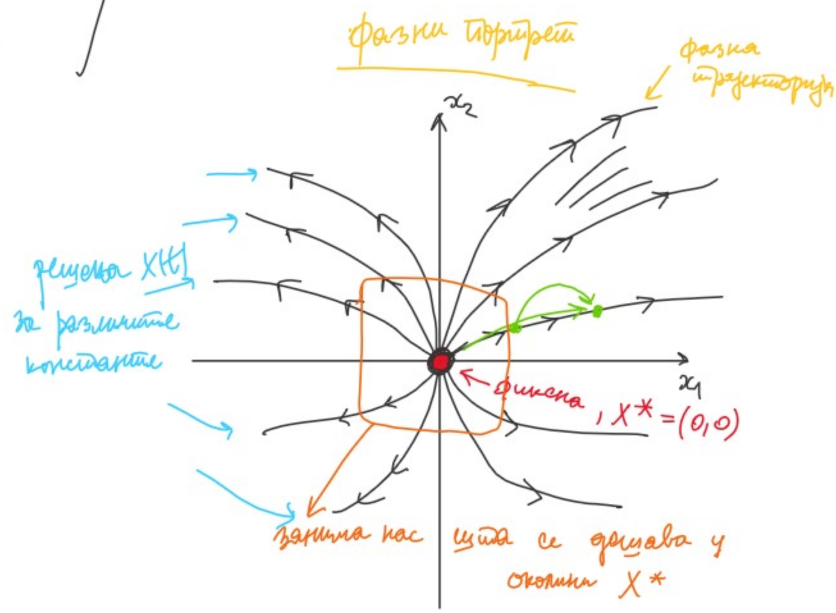
$$X(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad X'(t)=\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}$$

$X' = A\cdot X \rightarrow$ хомогена линеарна ДЈ (систем) (динамички систем)

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = A\cdot\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

решења $\underbrace{X(t)}_{(c\in\mathbb{R}^2)} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

$A\in M_2(\mathbb{R}) \Rightarrow$ са конст. коеф. (ЛСДЈКК)



Еквилибријуми : $X'=0$

ЛСДЈКК: $A\cdot X = 0 \Rightarrow$ решење је век. потпростор од $\mathbb{R}^2 \Rightarrow$
- тачка $(0,0)$
- права кроз $(0,0)$
- цело $\mathbb{R}^2$

https://aeb019.hosted.uark.edu/pplane.html

1. Скицирати фазни портрет динамичког система $X' = AX$, ако је:

а) $A=\begin{bmatrix}-1 & 2\\ 0 & -3\end{bmatrix}$; б) $A=\begin{bmatrix}-1 & 3\\ 5 & -3\end{bmatrix}$; в) $A=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$;

г) $A=\begin{bmatrix}-2 & -5\\ 2 & 2\end{bmatrix}$; д) $A=\begin{bmatrix}-1 & -1\\ 2 & -3\end{bmatrix}$; ђ) $A=\begin{bmatrix}5 & 0\\ 0 & 0\end{bmatrix}$.

$E=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = Id = I$ јединична матрица

Линк ка материјалима из курса ДЈБ одакле је узет овај задатак. Као што смо помињали, не морате решавати једначину (као у материјалима), ако знате са предавања како фазни портрет треба да изгледа.
https://tinyurl.com/fazniPortreti1

а) $\det(A-\lambda E)=0$, $X^*=(0,0)$

$$\det\left(\begin{bmatrix}-1-\lambda & 2\\ 0 & -3-\lambda\end{bmatrix}\right) = (1+\lambda)(3+\lambda) - 0 = 0$$

$\lambda_1=-1,\ \lambda_2=-3$

$$\det\left(\begin{bmatrix} & \\ 0 & -3-\lambda \end{bmatrix}\right) = (1+\lambda)(3+\lambda) - 0 = 0$$

$$\left.\begin{matrix} \underline{\lambda_1 = -1}, \ \underline{\lambda_2 = -3} \\ -1, -3 < 0 \end{matrix}\right\} \text{стабилни чвор}$$

$P^{-1} \backslash X' = AX$

$A = PDP^{-1}$

$$\Rightarrow \underbrace{P^{-1} \cdot X'}_{} = D\underbrace{P^{-1}X}_{}$$

$$Y' = DY$$

смена: $Y = P^{-1} \cdot X \Rightarrow Y' = P^{-1} \cdot X'$

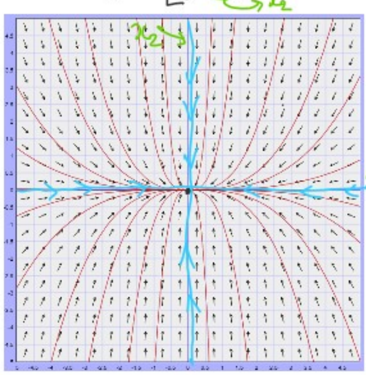
D: - дијагонална $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

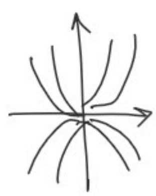
- $\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$ → комплексне сопс. вр. $\alpha \pm i\beta$

- $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ → 1 Жорданов блок

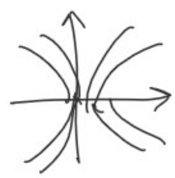
$$Y' = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} Y$$

x_1, x_2





?



?

$|\lambda_1| < |\lambda_2|$ → држе уз x_1-осу

→ овде су осе $\underbrace{\text{сопс. вектори}}_{\text{иду дуж оса}}$ од $\begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$

$A = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}$ → ове осе иду у базу сопс. вектора

$\lambda_1 = -1$: $\quad (A - \lambda_1 E)\underbrace{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}}_{= v} = 0 \qquad \begin{bmatrix} 0 & 2 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0$

$2\beta = 0 \Rightarrow \beta = 0 \Rightarrow v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\cancel{-2\beta = 0}$

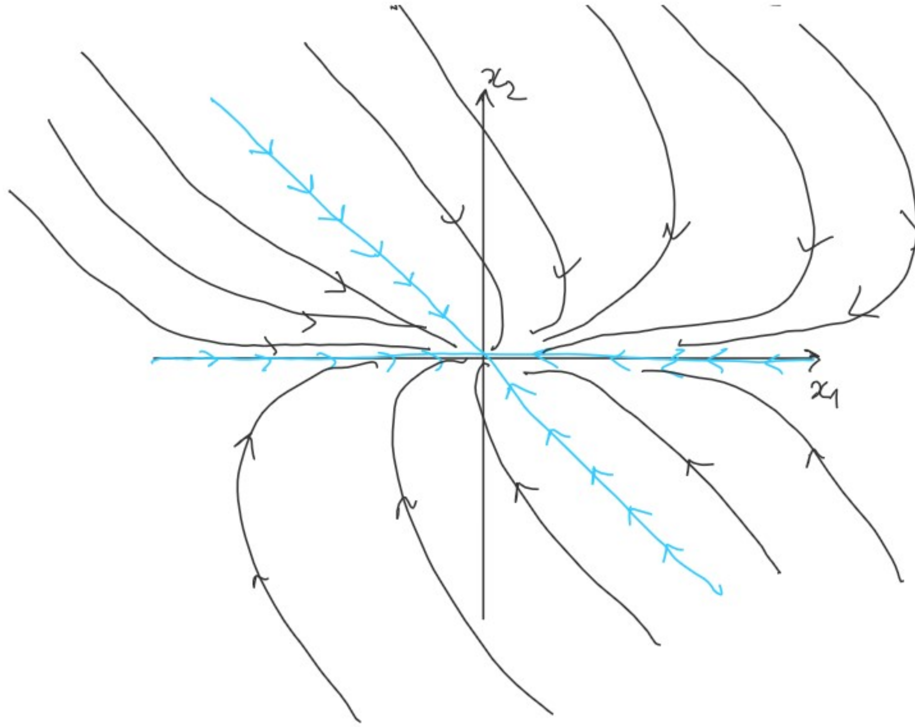
$\lambda_2 = -3$: $\quad (A + 3E)v = 0 \qquad \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0$

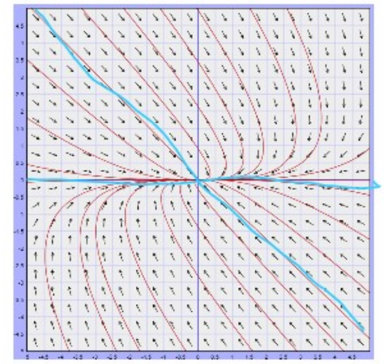
$2\alpha + 2\beta = 0 \Rightarrow v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\cancel{0 = 0}$

$2\alpha + 2\beta = 0 \Rightarrow v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\cancel{0 = 0}$





да ли се створи рефлектору? $P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $\det P = 1 > 0 \rightarrow$ нема рефл.

б) $A = \begin{bmatrix} -1 & 3 \\ 5 & -3 \end{bmatrix}$, $X^* = (0,0)$

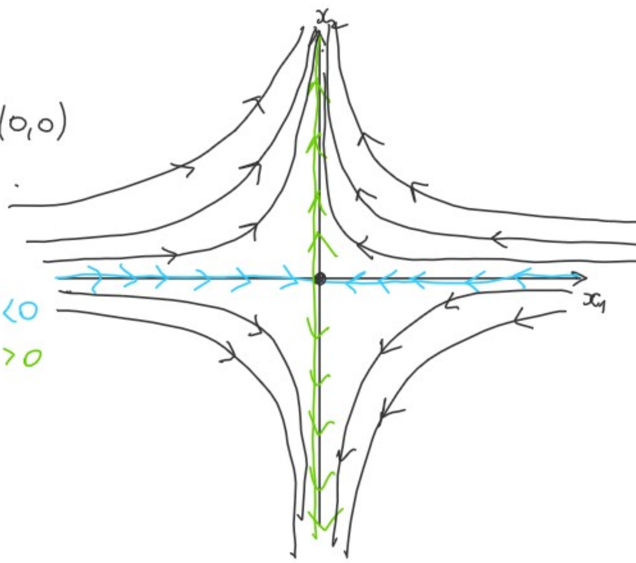
$\lambda_1 = -6$, $\lambda_2 = 2$

$D = \begin{bmatrix} -6 & 0 \\ 0 & 2 \end{bmatrix}$

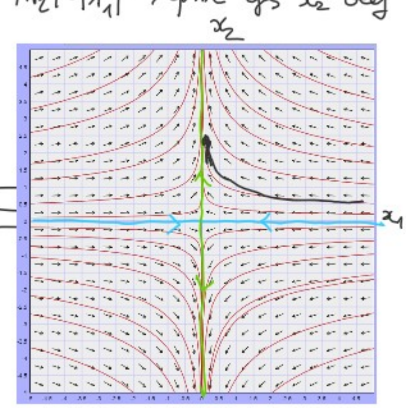
седло

$-6 < 0$

$2 > 0$

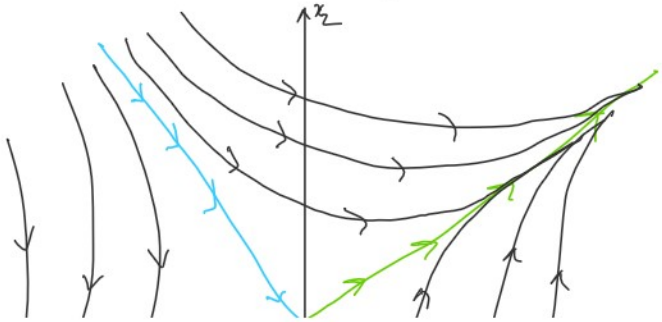


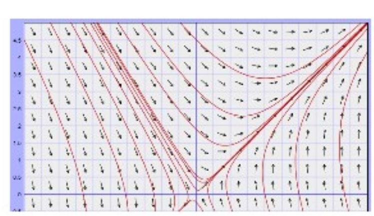
$|\lambda_2| < |\lambda_1| \rightarrow$ брже уз x_2 осу

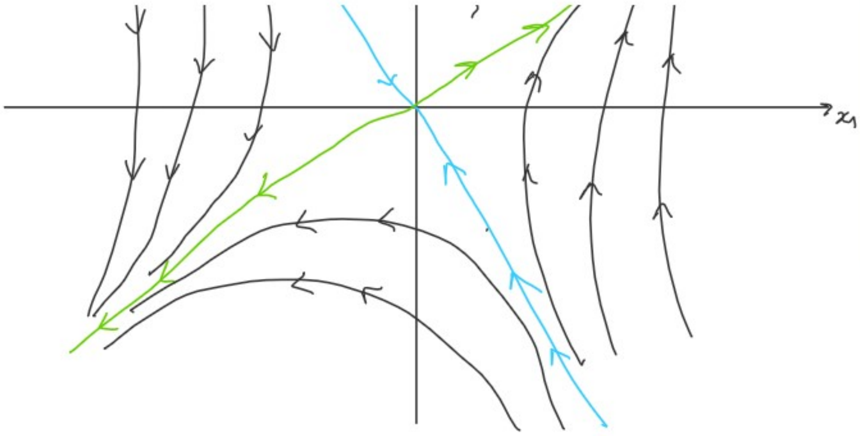


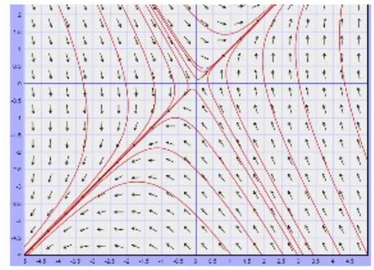
сопс. век: $\lambda_1 = -6 : v_1 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$, $P = \begin{bmatrix} 3 & 1 \\ -5 & 1 \end{bmatrix}$, $\det P = 3 + 5 = 8 > 0 \rightarrow$ нема рефл.

$\lambda_2 = 2 : v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$





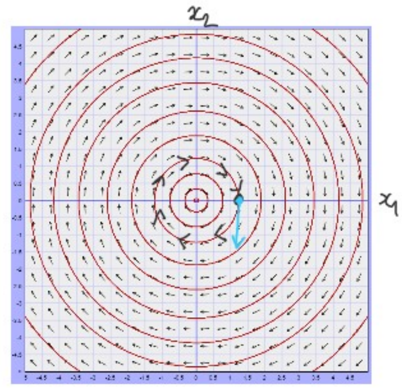




в) $A=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$ $\qquad \alpha \pm i\beta \ (\beta \neq 0)$, $X^*=(0,0)$

$\lambda_{1,2}=\pm i$ $\qquad \underline{\alpha=0} \to$ центар

↳ концентрични кругови



На коју страну су оријентисани?

$X'=AX$, $\left.\begin{matrix}x_1=1\\ x_2=0\end{matrix}\right\} \Rightarrow X'=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}=\begin{bmatrix}0\\ -1\end{bmatrix}$

$\sqrt{\ } A=PDP^{-1}$, $\lambda_{1,2}=\alpha \pm i\beta$ $\qquad P=?$

$D=\begin{bmatrix}\alpha & \beta\\ -\beta & \alpha\end{bmatrix}$

$\underbrace{\lambda_1=i}_{0+1\cdot i}$ $\qquad D=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$, $P=E=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $\det P=1>0$

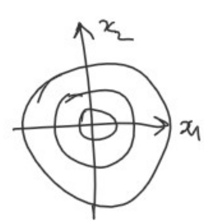
онда кругови у истом смеру

$A=PDP^{-1}=D$

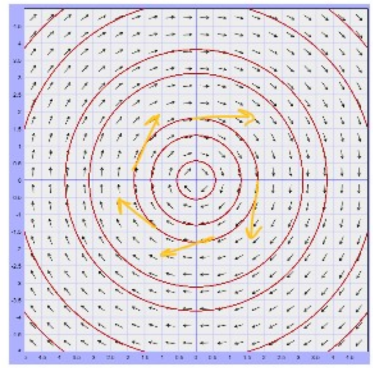
г) $A=\begin{bmatrix}-2 & -5\\ 2 & 2\end{bmatrix}$, $X^*=(0,0)$

$\lambda_{1,2}=\pm i\sqrt{6}$

$(\alpha=0)$ ↳ центар



$\lambda_1 = i\sqrt{6}$: $\quad D = \begin{bmatrix} 0 & \sqrt{6} \\ -\sqrt{6} & 0 \end{bmatrix} \xrightarrow{\text{за } D}$



$A = PDP^{-1}$, $P = \begin{bmatrix} Re(v)\downarrow & Im(v)\downarrow \end{bmatrix}$

$v \to$ комплексни сопс. век. за λ_1

$(A-\lambda_1 E)v = 0$

$$\begin{bmatrix} -2-i\sqrt{6} & -5 \\ 2 & 2-i\sqrt{6} \end{bmatrix} \cdot v = 0, \quad v = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$2^2 - (i\sqrt{6})^2 = 4+6 = 10$

$(-2-i\sqrt{6})\alpha - 5\beta = 0$

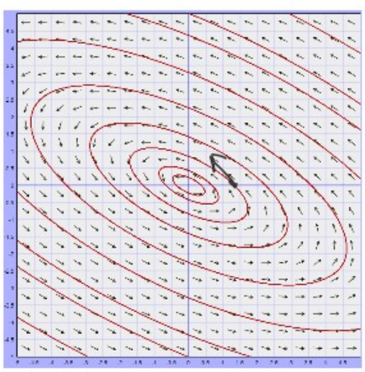
$2\alpha + (2-i\sqrt{6})\beta = 0 \; / \cdot (2+i\sqrt{6}) \longrightarrow (4+2i\sqrt{6})\alpha + 10\beta = 0 \; / \cdot (-\tfrac{1}{2}) \to (-2-i\sqrt{6})\alpha - 5\beta = 0$

$\beta = -\dfrac{2+\sqrt{6}i}{5} \cdot \alpha$, $\quad \alpha = -5$, $\beta = 2+i\sqrt{6}$ $\quad v = \begin{bmatrix} -5 \\ 2+i\sqrt{6} \end{bmatrix}$

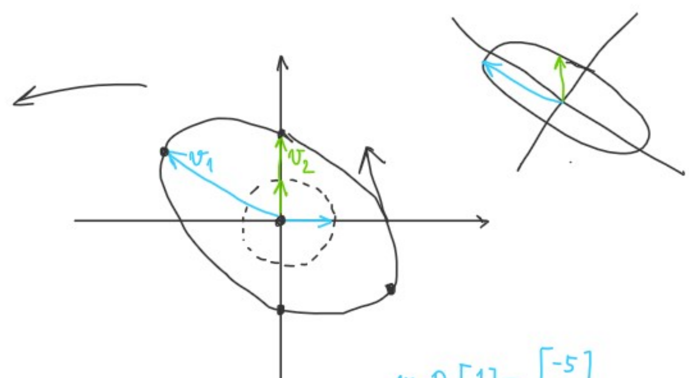
$Re(v) = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$, $Im(v) = \begin{bmatrix} 0 \\ \sqrt{6} \end{bmatrix}$

$\Rightarrow P = \begin{bmatrix} -5 & 0 \\ 2 & \sqrt{6} \end{bmatrix}$

$\det P = -5\sqrt{6} < 0$
мења се смер траекторија



$X' = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$
$X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$



$v_1 = P \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$

$v_2 = P \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{6} \end{bmatrix}$

б) $A = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$, $\quad X^* = ?$ $\quad \det(A) = 0$

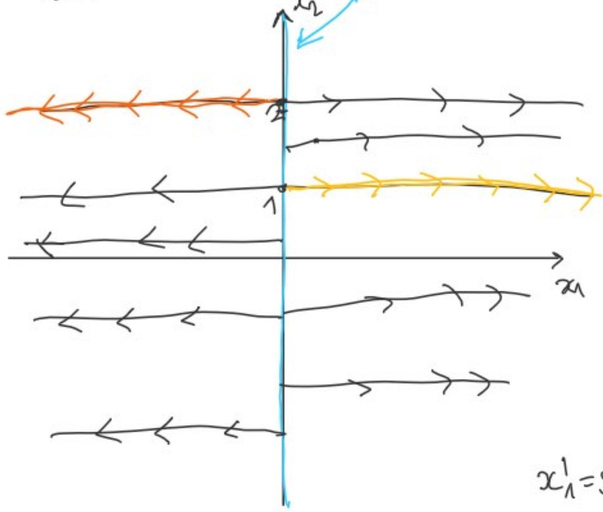
$\lambda_1 = 5 > 0$
$\lambda_2 = 0$

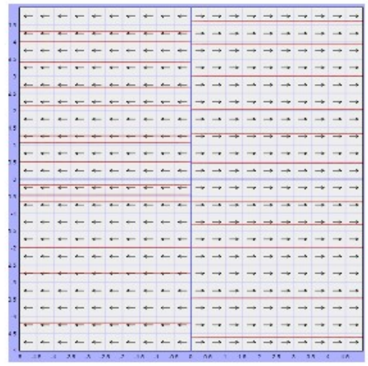
$$AX = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Bigg\} \Rightarrow \boxed{X^* = (0, \lambda)}, \lambda \in \mathbb{R}$$

$5x_1 = 0 \Rightarrow x_1 = 0$
$0 = 0$

(неизоловани)

$x_2 = 0$





$$\left.\begin{array}{l} x_1' = 5x_1 \\ x_2' = 0 \end{array}\right\} \begin{array}{l} x_1 = c_1 e^{5t} \\ x_2 = c_2 \end{array}, \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \in \mathbb{R}^2$$

$$\left.\begin{array}{l} c_1 = 1 \\ c_2 = 1 \end{array}\right\} \begin{array}{l} x_1 = e^{5t} \\ x_2 = 1 \end{array} \quad t\uparrow \Rightarrow x_1 \uparrow$$

$$\left.\begin{array}{l} c_1 = -2 \\ c_2 = 2 \end{array}\right\} \begin{array}{l} x_1 = -2e^{5t} \\ x_2 = 2 \end{array} \quad t\uparrow \Rightarrow x_1 \downarrow$$

Д) $A = \begin{bmatrix} -1 & -1 \\ 2 & -3 \end{bmatrix}$

$\lambda_{1,2} = -2 \pm i \quad (\alpha \neq 0)$

$-2 < 0$ → стабилна спирала

$X^* = (0,0)$

$\det A \neq 0$

$A = PDP^{-1}$

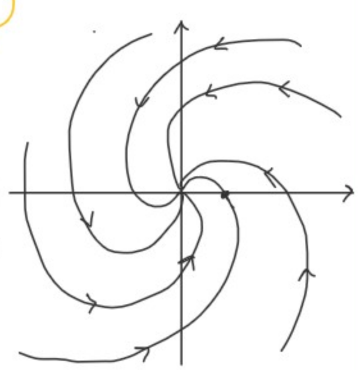
$1 > 0$

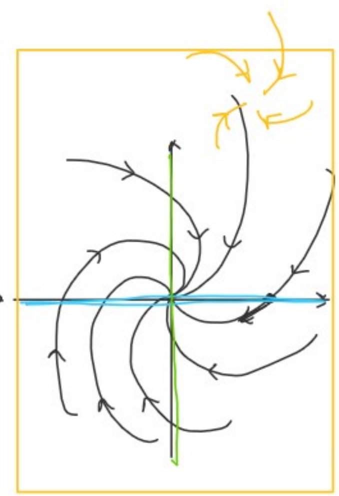
$D = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix}$

$\lambda = \alpha \pm i\beta \rightarrow D = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$

$P = [\mathrm{Re}(v)\downarrow \quad \mathrm{Im}(v)\downarrow]$

негативни смер



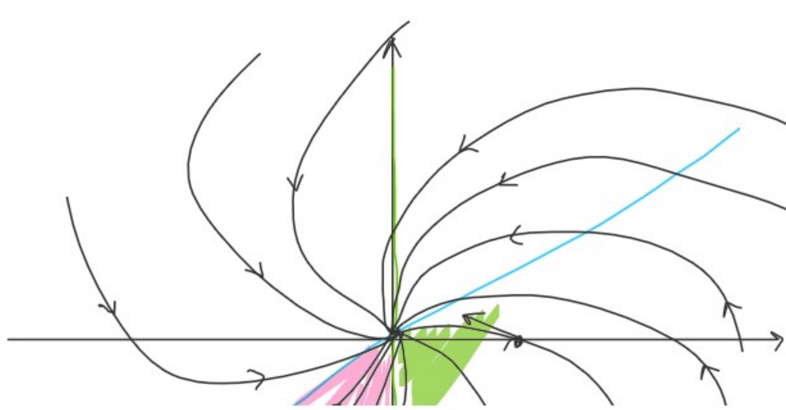


$$\begin{bmatrix} -1-(-2+i) & -1 \\ 2 & -3-(-2+i) \end{bmatrix} v = 0$$

$(1-i)\alpha - \beta = 0 \Rightarrow \beta = (1-i)\alpha$

$v = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$

$P = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad \det P = -1 < 0$



$A \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = X'$