4) Формирати низ итерација из Пикарове T за проблем: $x' = \frac{x}{t}$, $x(t_0) = x_0$, $t_0 > 0$

$$F(x,t) = \frac{x}{t}$$

$$\underbrace{\mathbf{x}_0(t) \equiv \mathbf{x}_0,} \quad \mathbf{x}_{n+1}(t) := \mathbf{x}_0 + \int_{t_0}^{t} F(\mathbf{x}_n(s), s)\, ds,$$

$$x_1(t) = x_0 + \int_{t_0}^{t} \frac{\overset{x_0}{x_0(s)}}{s}\, ds = x_0 + (\ln(t) - \ln(t_0)) \cdot x_0$$

$$x_2(t) = x_0 + \int_{t_0}^{t} \frac{x_1(s)}{s}\, ds = x_0 + \int_{t_0}^{t} \frac{\overset{\text{конст.}}{x_0(1-\ln(t_0))} + \ln(s)}{s}\, ds = x_0 + x_0 \cdot (1-\ln(t_0)) \cdot (\ln(s))\Big|_{t_0}^{t}$$

$$+ x_0 \cdot \int_{t_0}^{t} \frac{\ln s}{s}\, ds \qquad \ln s = u$$

$$= x_0 + x_0(1-\ln(t_0)) \cdot (\ln(t) - \ln(t_0)) + x_0 \cdot \left(\frac{\ln^2 s}{2}\right)\Big|_{t_0}^{t} =$$

$$= x_0 + x_0(1-\ln(t_0)) \cdot (\ln(t) - \ln(t_0)) + \frac{x_0}{2}(\ln^2 t - \ln^2 t_0)$$

$$x_3(t) = x_0 + \int_{t_0}^{t} \frac{x_2(s)}{s}\, ds = x_0 + \int_{t_0}^{t} \frac{x_0 - \ln(t_0) \cdot x_0 \cdot (1-\ln(t_0)) - \frac{x_0}{2}\ln^2 t_0}{s}\, ds + \int_{t_0}^{t} \frac{x_0(1-\ln(t_0)) \cdot \ln(s)}{s}\, ds$$

$$+ \int_{t_0}^{t} \frac{x_0}{2} \cdot \frac{\ln^2(s)}{s}\, ds =$$

$$\int \frac{\ln^n s}{s}\, ds = \int_{\uparrow\, u = \ln s} u^n\, du = \frac{u^{n+1}}{n+1} + C = \frac{(\ln s)^{n+1}}{n+1} + C$$

$$= x_0 + \left(x_0 - \ln(t_0) \cdot x_0(1-\ln(t_0)) - \frac{x_0}{2}\ln^2(t_0)\right) \cdot (\ln(t) - \ln(t_0)) + x_0(1-\ln(t_0)) \cdot \frac{1}{2}(\ln^2(t) - \ln^2(t_0))$$

$$+ \frac{x_0}{2} \cdot \frac{1}{3} \cdot (\ln^3 t - \ln^3 t_0)$$

$$= \left(x_0 - x_0 \ln^{1} t_0 + \frac{1}{2} x_0 \ln^{2} t_0 - \frac{1}{6} x_0 \ln^{3} t_0\right) + \ln^{1} t \left(x_0 - x_0 \ln t_0 + \frac{1}{2} x_0 \ln^2 t_0\right) + \frac{\ln^{2} t}{2}(x_0 - x_0 \ln t_0) + \frac{\ln^{3} t}{6}(x_0)$$

индукцијом: $$x_n(t) = x_0 \cdot \sum_{k=0}^{n} \frac{\ln^{k} t}{k!} \cdot \underbrace{\boxed{\sum_{m=0}^{n-k} \frac{(-\ln t_0)^{m}}{m!}}}_{= a_{n-k}}$$

$k=0 \rightarrow m=3$

$k=1 \rightarrow m=2$

$$\int^{t} \ln^k s \ldots = \frac{1}{\ldots}(\ln^{k+1} t - \ln^{k+1} \ldots)$$

$k=0 \to m=3$

$k=1 \to m=2$

$k=2 \to m=1$

$k=3 \to m=0$

$$\int_{t_0}^{t} \frac{\ln^k s}{s}\, ds = \frac{1}{k+1}\left(\ln^{k+1} t - \ln^{k+1} t_0\right)$$

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} \frac{x_n(s)}{s}\, ds = x_0 + x_0 \cdot \int_{t_0}^{t} \frac{\sum_{k=0}^{n} \frac{\ln^k s}{k!} \cdot a_{n-k}}{s}\, ds = x_0 + x_0 \cdot \sum_{k=0}^{n} \underbrace{\frac{a_{n-k}}{k!} \cdot \frac{1}{k+1}} \cdot \left(\ln^{k+1} t - \ln^{k+1} t_0\right)$$

$$= x_0 + x_0 \cdot \sum_{k=1}^{n+1} \frac{a_{n+1-k}}{k!} \cdot \left(\ln^k t - \ln^k t_0\right)$$

уз $\ln^k t$ за $1 \le k \le n+1$: $\frac{a_{n+1-k}}{k!} \cdot x_0$ ✓

слободан члан, $k=0$: $(\ln^0 t)$

$$x_0 + x_0 \cdot \sum_{k=1}^{n+1} \frac{a_{n+1-k}}{k!} \cdot \left(-\ln^k t_0\right) = x_0 - x_0 \cdot \sum_{k=1}^{n+1} \frac{\ln^k t_0}{k!} \cdot \sum_{m=0}^{n+1-k} \frac{(-\ln t_0)^m}{m!}$$

уз $\ln^p t_0$: $p=0$ ✓

$$n+1 \geq p \geq 1: \quad \sum_{q=1}^{p} \frac{1}{q!} \cdot \frac{(-1)^{p-q}}{(p-q)!} = \sum_{q=1}^{p} \frac{(-1)^{p-q}}{p!} \cdot \binom{p}{q} =$$

$[k=q,\ m=p-q]$

$$(x+1)^p = \sum_{q=0}^{p} x^q \cdot \binom{p}{q} = 1 + \sum_{q=1}^{p} x^q \cdot \binom{p}{q}$$

$$x=-1: \quad 0 = (1-1)^p = 1 + \sum_{q=1}^{p} (-1)^q \binom{p}{q}$$

$$\Rightarrow \sum_{q=1}^{p} (-1)^q \binom{p}{q} = -1$$

$$= \frac{(-1)^p}{p!} \cdot \sum_{q=1}^{p} (-1)^q \cdot \binom{p}{q} = \frac{(-1)^p}{p!} \cdot (-1)$$

$$x_0 - x_0 \cdot \sum_{p=1}^{n+1} \ln^p t_0 \cdot \frac{(-1)^p \cdot (-1)}{p!} = x_0 + \sum_{p=1}^{n+1} \frac{(-\ln t_0)^p}{p!} \cdot x_0$$

$$= x_0 \cdot \sum_{p=0}^{n+1} \frac{(-\ln t_0)^p}{p!} \quad ✓$$

Чему тежи $x_n(t)$?

$$e^x = \sum_{l=0}^{\infty} \frac{x^l}{l!}$$

$$x_\infty(t) = x_0 \cdot \underbrace{\sum_{k=0}^{\infty} \frac{\ln^k t}{k!}}_{\ln t} \cdot \underbrace{\left(\sum_{m=0}^{\infty} \frac{(-\ln t_0)^m}{m!}\right)}_{e^{-\ln t_0}} =$$

$$\underbrace{\quad k=0 \quad \cdots \quad}_{e^{\ln t}} \qquad \underbrace{\quad m=0 \quad}_{e^{-\ln t_0}}$$

$$= x_0 \cdot t \cdot \frac{1}{t_0} = \frac{x_0}{t_0} \cdot t$$

$$x' = \left(\frac{x_0}{t_0} t\right)' = \frac{x_0}{t_0} = \frac{\frac{x_0}{t_0} t}{t} = \frac{x}{t} \quad , \quad x(t_0) = \frac{x_0}{t_0} \cdot t_0 = x_0 .$$