① Нека је $[a_{ij}]_{i,j=1}^{n} = A \in M_n(\mathbb{R})$, $a_{ij} \geq 0$ $\forall i \neq j$. Нека је $B = e^A = [b_{ij}]_{i,j=1}^{n} \in M_n(\mathbb{R})$. Доказати да је $b_{ij} \geq 0$ $\forall i,j$.

$$A = \begin{bmatrix} & & \geq 0 \\ & ? & \\ \geq 0 & & \end{bmatrix} \rightsquigarrow B = e^A = \begin{bmatrix} \geq 0 \end{bmatrix}$$

$$A = \begin{bmatrix} \geq 0 \end{bmatrix} \rightsquigarrow A^k = \begin{bmatrix} \geq 0 \end{bmatrix} \rightsquigarrow B = e^A = \begin{bmatrix} \geq 0 \end{bmatrix}$$

идеја: $A = A_1 + E_1$

A_1 – ел. ≥ 0

e^{E_1} – ел. ≥ 0 и $A_1 E_1 = E_1 A_1$ $\Big\}$ E_1 – пропорционална јединичној

$$A = \begin{bmatrix} -2 & & \\ & 4 & \\ & & -6 \end{bmatrix}$$

$M = |-6| = 6$

$A_1 = A + 6 \cdot E$

$$A_1 = \begin{bmatrix} 4 & & \\ & 10 & \\ & & 0 \end{bmatrix}$$

$$E_1 = -E \cdot \underbrace{\max_{1 \leq i \leq n} |a_{ii}|}_{= M}$$

$$A_1 = A + ME = A - E_1 \quad (A = A_1 + E_1)$$

гл. диј.

$$A_1 = [c_{ij}]_{i,j=1}^{n}$$

$$c_{ij} = \begin{cases} \underbrace{a_{ij}}_{\geq 0}, & i \neq j \\ \underbrace{a_{ii} + M}_{\geq 0}, & i = j \end{cases} \geq 0$$

$$(\forall i) \quad a_{ii} + M = a_{ii} + \max_{1 \leq j \leq n} |a_{jj}| \underset{j=i}{\geq} a_{ii} + |a_{ii}| \geq 0$$

$$A_1 E_1 = A_1 \cdot (-ME) = -M \cdot (A_1 E) = -M \cdot A_1 = -M \cdot (E A_1) = (-ME) \cdot A_1 = E_1 \cdot A_1$$

$$B = e^A = e^{A_1 + E_1} \overset{(2)}{=} e^{A_1} \cdot e^{E_1}$$

e^{A_1}: ел. ≥ 0, јер A_1 има ел. ≥ 0

e^{E_1}: ел. ≥ 0 ($e^{-M} \geq 0$)

$\Big\} \Rightarrow b_{ij} \geq 0 \ (\forall i,j)$

$$e^{E_1} = \exp\left(\begin{bmatrix}-M & & \\ & \ddots & \\ & & -M\end{bmatrix}\right) = \begin{bmatrix} e^{-M} & & \\ & \ddots & \\ & & e^{-M}\end{bmatrix}$$

Фундаментални систем (скуп) решења : скуп n независних решења $\varphi_1(t), \dots, \varphi_n(t)$ система $X'(t) = A(t)\cdot X(t)$

$$\Phi(t) = \left[\varphi_1(t)\downarrow \;\dots\; \varphi_n(t)\downarrow\right]$$ — фундаментална матрица

$$W(t) = \det \Phi(t)$$ — Вронскијан

⊛ Нека матрица је фундаментална за систем $X' = A(t)X$ $\iff$ $\Phi'(t) = A(t)\cdot\Phi(t)$ и $W(t) \neq 0$. независна

$$\left[\varphi_1'(t) \dots \varphi_n'(t)\right] = A(t)\cdot\left[\varphi_1(t) \dots \varphi_n(t)\right]$$

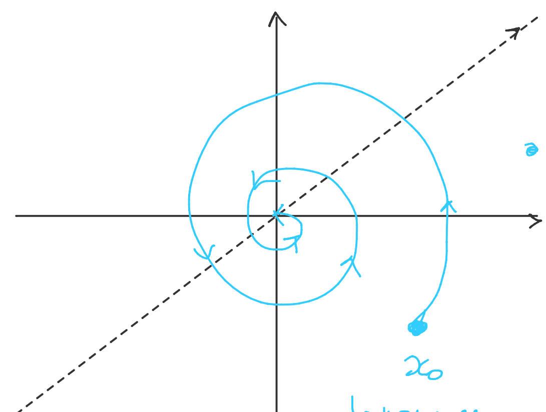
$$\forall i \in \{1, \dots, n\} \quad \varphi_i'(t) = A(t)\cdot\varphi_i(t)$$

⊛ $\Phi(t) \rightsquigarrow A(t) = \Phi'(t)\cdot\Phi(t)^{-1}$ (уназад)

$X' = AX \rightsquigarrow X = e^{tA}\cdot \underset{\sim}{c}$

$A \in M_n(\mathbb{R})$ $\quad c = X(0) = x_0$

$X : \underset{\Uparrow\, t}{\mathbb{R}} \to \mathbb{R}^n$



x_0

↳ како се свака тачка x_0 креће у простору

$$x_0 \longmapsto \underbrace{e^{tA}}\cdot x_0$$

↳ кроз време

$\phi_t : \mathbb{R}^n \to \mathbb{R}^n$

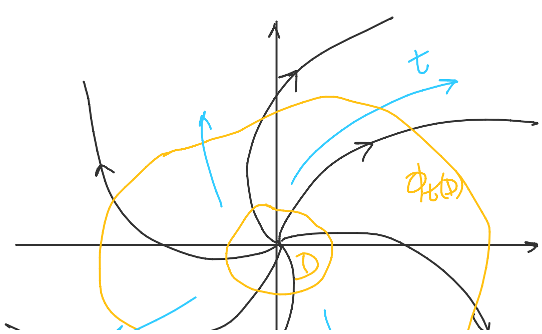
$\phi_t(x_0) = e^{tA}\cdot x_0$ } кретање (ток) система

$\phi_0 = id$

$\boxed{T}$ (Лиувил) Ток ϕ_t система $X' = AX$ мења запремину (мерење) области $D \subseteq \mathbb{R}^n$ као

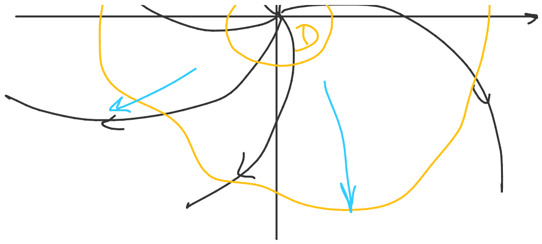
$$Vol(\phi_t(D)) = e^{t\cdot trA}\cdot Vol(D)$$

пр.



$Vol(\phi_t(D)) > Vol(D)$ — повећава запремину

$t > 0$



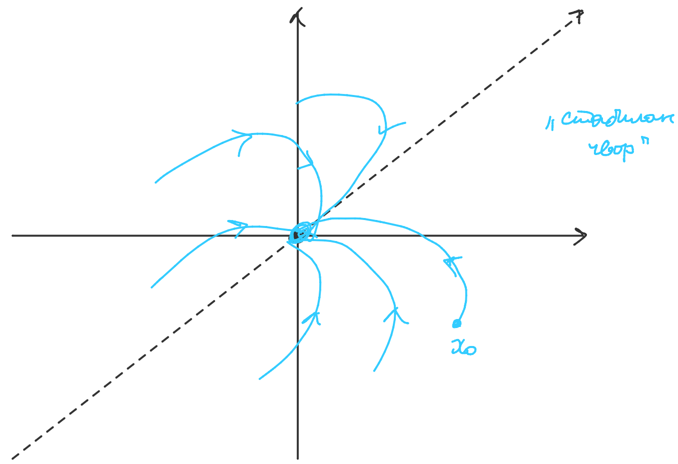
$t>0$

- смањује $tr(A)<0$
- чува $tr(A)=0$
- повећава $tr(A)>0$

② $X' = \begin{bmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3\end{bmatrix} X$, $\underline{\lambda_3<\lambda_2<\lambda_1<0}$. Како се креће произ. решење? Како систем мења запремину?

$$\phi_t(x_0) = e^{tA}\cdot x_0 = \begin{bmatrix}e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & e^{\lambda_3 t}\end{bmatrix}\cdot x_0$$

$$\lim_{t\to+\infty} \phi_t(x_0) = \begin{bmatrix}0\\0\\0\end{bmatrix}$$



$tr A = \lambda_1+\lambda_2+\lambda_3 < 0 \Rightarrow$ систем смањује запремину

③ Доказати да је за ЛСДЈКК $X'=AX$ фундаментална матрица дата са $\Phi(t)=e^{tA}$.

1) $\Phi' = A\Phi$?

$$\Phi'(t) = (e^{tA})' \overset{(5)}{=} A\cdot e^{tA} = A\cdot\Phi(t) \quad ✓$$

2) $W(t) = \det(e^{tA}) \overset{(6)}{=} e^{tr(t\cdot A)} > 0 \Rightarrow W(t)\neq 0$

нпр. $e^{tA} = \begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}$ независна решења система

<u>Решавање ЛСДЈКК у општем случају</u>

$X'=AX$

$A = P\,D\,P^{-1}$ у Жордановој нормалној форми

$$X(t) = \underline{e^{tA}} \cdot c = e^{t(PDP^{-1})} \cdot c \overset{(7)}{=} P \cdot \underline{e^{tD}} \cdot P^{-1} \cdot c \qquad , c \in \mathbb{R}^n$$

e^{tD} - знамо да одредимо

P - матрица прелаframerска

A

$D = ?$

$P = ?$ $(P^{-1} = ?)$

$e^{tD} = ?$

(4)

$x_1' = x_1 - x_2 + x_3$

$x_2' = x_1 + x_2 - x_3$

$x_3' = 2x_1 - x_2$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad , X' = AX \quad , A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

$-$ $+$

$\det(A - \lambda E) = 0$

$$0 = \det\left(\begin{bmatrix} 1-\lambda & -1 & 1 \\ 1 & 1-\lambda & -1 \\ 2 & -1 & -\lambda \end{bmatrix}\right) = -2\cdot(1-\lambda)\cdot 1 - 1\cdot(-1)\cdot(-\lambda) - (1-\lambda)\cdot(-1)^2 + (1-\lambda)^2(-\lambda) + (-1)^2\cdot 2 + 1^2\cdot(-1) =$$

$$= \cancel{-2} + 2\lambda \cancel{-\lambda} - 1 + \cancel{\lambda} - \lambda(1 - 2\lambda + \lambda^2) + \cancel{2} - 1 =$$

$$= 2\lambda - 2 - \lambda + 2\lambda^2 - \lambda^3 = -\lambda^3 + 2\lambda^2 + \lambda - 2 =$$

$\lambda = 1$

$$= (\lambda - 1)(-\lambda^2 + \lambda + 2) =$$

$$= (\lambda - 1)(\lambda + 1)(2 - \lambda)$$

$$\left.\begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = -1 \\ \lambda_3 = 2 \end{array}\right\}$$ сопс. вр. су реалне и различите $\rightarrow$ 3 сопс. век.

$\lambda_1 = 1$: $(A - \lambda_1 E)\gamma_1 = 0$

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 2 & -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \qquad \left.\begin{array}{r} -b + c = 0 \\ a \quad - c = 0 \\ 2a - b - c = 0 \end{array}\right\}$$

$$\left.\begin{array}{r} -b + c = 0 \\ a - b \quad = 0 \\ 2a - 2b = 0 \end{array}\right\} \quad a = b \quad c = b = a \quad \Rightarrow \gamma_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\lambda_2 = -1$: $(A - \lambda_2 E)\gamma_2 = 0$

$\lambda_2 = -1: \quad (A - \lambda_2 E)\gamma_2 = 0$

$$(A+E)\gamma_2 = 0 \quad \therefore \quad \gamma_2 = \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix}$$

$$\lambda_3 = 2: \quad (A - \lambda_3 E)\gamma_3 = 0 \quad \Rightarrow \quad \gamma_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} \gamma_1 \downarrow & \gamma_2 \downarrow & \gamma_3 \downarrow \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -3 & 0 \\ 1 & -5 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & & \\ & -1 & \\ & & 2 \end{bmatrix} \Rightarrow e^{tD} = \begin{bmatrix} e^{t} & & \\ & e^{-t} & \\ & & e^{2t} \end{bmatrix}$$

$P^{-1} = ?$

$$P^{-1} = \frac{1}{\det P} \cdot \operatorname{Adj} P = \frac{1}{-6} \cdot \begin{bmatrix} +\begin{vmatrix} -3 & 0 \\ -5 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & -3 \\ 1 & -5 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ -5 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & -5 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ -3 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} \end{bmatrix}^T = \frac{1}{-6} \begin{bmatrix} -3 & -1 & -2 \\ -6 & 0 & 6 \\ 3 & 1 & -4 \end{bmatrix}^T = \frac{1}{6} \begin{bmatrix} 3 & 6 & -3 \\ 1 & 0 & -1 \\ 2 & -6 & 4 \end{bmatrix}$$

$$\det P = -3 + 0 - 5 + 3 - 0 - 1 = -6$$

$$\text{OP}: \quad X(t) = P e^{tD} P^{-1} \cdot c = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -3 & 0 \\ 1 & -5 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{t} & & \\ & e^{-t} & \\ & & e^{2t} \end{bmatrix} \cdot \underbrace{\frac{1}{6} \cdot \begin{bmatrix} 3 & 6 & -3 \\ 1 & 0 & -1 \\ 2 & -6 & 4 \end{bmatrix} \cdot c}_{= c_1 \in \mathbb{R}^3}, \quad c \in \mathbb{R}^3$$

$$= P \cdot e^{tD} \cdot c_1, \quad c_1 \in \mathbb{R}^3$$

(* може и без P^{-1} ако не трябва!)

(5) $x_1' = -3x_1$

$x_2' = \quad 3x_2 - 2x_3$

$x_3' = \quad x_2 + x_3$

$$A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\det(A - \lambda E) = 0 \Rightarrow \ldots \quad -(\lambda+3)(\lambda^2 - 4\lambda + 5) = 0$$

$\lambda_1 = -3$

$\lambda_2 = 2 + i$

$\lambda_3 = 2 - i$ — компл. конј.

$\lambda_1 = -3$, $\quad (A - \lambda_1 E)\gamma_1 = 0 \ldots \Rightarrow \gamma_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

канон. собс. век.

$\lambda_2 = 2 + i$: $\quad (A - \lambda_2 E)\gamma_2 = 0$

(За λ_3 не трябва пр $\overline{\lambda_3} = \lambda_2 \Rightarrow \overline{\gamma_3} = \gamma_2$)

$$\begin{bmatrix} -3-(2+i) & 0 & 0 \\ 0 & 3-(2+i) & -2 \\ 0 & 1 & 1-(2+i) \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$(-5-i)a = 0 \quad \Rightarrow a = 0$

$(1-i)b - 2c = 0 \;\Rightarrow\; c = \frac{1-i}{2} \cdot b$

$b + (-1-i)c = 0 \;\big|\cdot(1-i)\; \} \quad (1-i)b - \underbrace{(1+i)(1-i)}_{-2}c = 0$

$$\gamma_2 = \begin{bmatrix} 0 \\ 1+i \\ 1 \end{bmatrix}$$

$b = 1+i$

$\Rightarrow c = 1$

$$\alpha + i\beta \longleftrightarrow \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

$$\lambda_2 = 2+i \longleftrightarrow \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

$\alpha = 2$

$\beta = 1$

$\lambda_1 = -3$

$$D = \begin{bmatrix} \boxed{-3} & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

$\lambda_2 = 2+i$

2×2

$$e^{tD} = \begin{bmatrix} e^{-3t} & 0 & 0 \\ 0 & & \\ 0 & \multicolumn{2}{c}{\boxed{\exp\left(t\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\right)}} \end{bmatrix} = \begin{bmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{2t}\cos t & e^{2t}\sin t \\ 0 & -e^{2t}\sin t & e^{2t}\cos t \end{bmatrix}$$

2×2

(1г) - дължно

$e^{2t} \cdot R_{(1 \cdot t)}$

Блок матр.:

$$D = \begin{bmatrix} \boxed{B_1} & \\ & \boxed{B_2} \end{bmatrix}$$

$$e^{tD} = \begin{bmatrix} \boxed{e^{tB_1}} & \\ & \boxed{e^{tB_2}} \end{bmatrix}$$

$$P = \begin{bmatrix} \gamma_1 \downarrow & \mathrm{Re}\,\gamma_2 \downarrow & \mathrm{Im}\,\gamma_2 \downarrow \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\gamma_2 = \begin{bmatrix} 0 \\ 1+i \\ 1 \end{bmatrix} \Rightarrow \mathrm{Re}\,\gamma_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathrm{Im}\,\gamma_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

За вкъщи: $P^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -2 \end{bmatrix}$

ОР: $x(t) = P \cdot e^{tD} \cdot c$, $c \in \mathbb{R}^3$

$\hookrightarrow P^{-1} \cdot \tilde{c}$