Експонент матрице

$X' = AX$ ЛСДЈКК

$A \in M_n(\mathbb{R})$ $X, X' \in \mathbb{R}^n$

$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \Rightarrow \begin{matrix} x_1' = a_{11}x_1 + \dots + a_{1n}x_n \\ \vdots \quad \ddots \\ x_n' = a_{n1}x_1 + \dots + a_{nn}x_n \end{matrix}$ $A = [a_{ij}]_{i,j=1}^n$

ОР је $X(t) = \underbrace{e^{tA}} \cdot c$, $c \in \mathbb{R}^n$ $c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

$\exp : M_n(\mathbb{R}) \to M_n(\mathbb{R})$, $\exp(A) = e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$

Тврђење 52. **(Својства експонента.)**

0 = нула матрица

$\mathbf{0}, \underbrace{\mathrm{Id}}_{E} \in M_n(\mathbb{R})$

(1) $e^{\mathbf{0}} = \mathrm{Id}$;
(2) $AB = BA \Rightarrow e^{A+B} = e^A e^B$;
(3) $AB = BA \Rightarrow Be^A = e^A B$;
(4) $e^A = \lim_{n\to\infty} \left(\mathrm{Id} + \frac{A}{n}\right)^n$;
(5) за $U = \mathbb{R}^n$, тј. $A \in M_n(\mathbb{R})$ важи $\frac{d}{dt}e^{tA} = e^{tA}A \boxed{= Ae^{tA}}$; $(3) \Rightarrow (A \cdot tA = tA^2 = tA \cdot A \Rightarrow Ae^{tA} = e^{tA}A)$
(6) за $U = \mathbb{R}^n$ важи $\det e^A = e^{\operatorname{tr} A}$;
(7) за $U = \mathbb{R}^n$ важи $e^{P^{-1}AP} = P^{-1}e^A P$.

(1) Решити систем ДЈ $X' = AX$, одређивањем e^{tA} у облику реда, ако је:

а) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ б) $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ в) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ г) $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, $a, b \in \mathbb{R}$

предавања (битно!)

а) $X(t) = \underbrace{e^{tA}}_{?} \cdot c$, $c \in \mathbb{R}^2$

$e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$

$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$A^3 = A^2 A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$\Big\}$ индукцијом: $A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$? ✓

Д: ✓

Б: ✓

Х: $A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

К: $A^{k+1} = A^k \cdot A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k+1 \\ 0 & 1 \end{bmatrix}$ ✓

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{t^k}{k!} & \sum_{k=0}^{\infty} \frac{t^k}{k!}\cdot k \\ 0 & \sum_{k=0}^{\infty} \frac{t^k}{k!} \end{bmatrix} = \begin{bmatrix} e^t & t\cdot e^t \\ 0 & e^t \end{bmatrix}$$

$$\sum_{k=0}^{\infty}\frac{t^k}{k!} = e^t \quad , \quad \underbrace{\sum_{k=0}^{\infty}\frac{t^k}{k!}\cdot k}_{} = \sum_{k=1}^{\infty}\frac{t^k}{(k-1)!} = t\cdot\sum_{k=1}^{\infty}\frac{t^{k-1}}{(k-1)!} = t\cdot\underbrace{\sum_{k=0}^{\infty}\frac{t^k}{k!}}_{} = t\cdot e^t$$

б) домаћи: пробајте исто као а)

$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

$A = \underbrace{\begin{bmatrix} \cdots \end{bmatrix}}_{D} + \underbrace{\begin{bmatrix} \cdots \end{bmatrix}}_{N}$

идеја: лако наћи e^{tD} и e^{tN}.

ОР: $X = \underline{e^{tA}}\cdot c$, $c \in \mathbb{R}^2$

$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$e^{tA} = e^{tD+tN} \overset{?}{=} e^{tD}\cdot e^{tN}$

(2) ? $tD\cdot tN = tN\cdot tD$?

$\Downarrow$

$e^{tA} = e^{tD}\cdot e^{tN}$

$tD\cdot tN = t^2\cdot D\cdot N = t^2\cdot\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = t^2\cdot\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\neq$

$tN\cdot tD = t^2\cdot N\cdot D = t^2\cdot\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = t^2\cdot\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$

не комутирају $\Rightarrow$ не можемо (2)

Пробамо другачије: $D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $N = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

(свака матрица комутира са E: $AE = EA = A$)

$D\cdot N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

(свака матрица комутира са E: $AE=EA=A$)

$$\left.\begin{array}{l} D\cdot N = \begin{bmatrix}1&0\\0&1\end{bmatrix}\cdot\begin{bmatrix}0&1\\0&1\end{bmatrix} = \begin{bmatrix}0&1\\0&1\end{bmatrix} \\[2ex] N\cdot D = \begin{bmatrix}0&1\\0&1\end{bmatrix}\cdot\begin{bmatrix}1&0\\0&1\end{bmatrix} = \begin{bmatrix}0&1\\0&1\end{bmatrix}\end{array}\right\} \Rightarrow \text{комутирају}$$

$$(2) \Rightarrow \quad e^{tA} = e^{tD+tN} = \underline{e^{tD}}\cdot\underline{e^{tN}}$$

$$e^{tD} = e^{tE} = \sum_{k=0}^{\infty}\frac{(tE)^k}{k!} = \sum_{k=0}^{\infty}\frac{t^k E}{k!} = E\cdot\underbrace{\sum_{k=0}^{\infty}\frac{t^k}{k!}}_{e^t} = e^t E = \begin{bmatrix}e^t&0\\0&e^t\end{bmatrix}$$

$E^k = E \quad (\forall k)$

$$e^{tN} = \sum_{k=0}^{\infty}\frac{(tN)^k}{k!} = \sum_{k=0}^{\infty}\frac{t^k\cdot N^k}{k!} = N\cdot\sum_{k=0}^{\infty}\frac{t^k}{k!} = e^t\cdot N = \begin{bmatrix}0&e^t\\0&e^t\end{bmatrix}$$

$$N^k? \quad N^2 = \begin{bmatrix}0&1\\0&1\end{bmatrix}\begin{bmatrix}0&1\\0&1\end{bmatrix} = \begin{bmatrix}0&1\\0&1\end{bmatrix} = N \Rightarrow N^k = N = \begin{bmatrix}0&1\\0&1\end{bmatrix}, (\forall k)$$

$$e^{tA} = e^{tD}\cdot e^{tN} = \underbrace{e^t\cdot E}\cdot\underbrace{e^t\cdot N} = e^{2t}E\cdot N = e^{2t}\cdot N = \begin{bmatrix}0&e^{2t}\\0&e^{2t}\end{bmatrix}.$$

Г) $A = \begin{bmatrix}a&b\\-b&a\end{bmatrix} \longleftrightarrow a+ib$ $\qquad e^{t(a+ib)} = e^{ta}(\cos(tb)+i\sin(tb))$

$$e^{tA} = \underbrace{e^{ta}}\cdot\underbrace{\begin{bmatrix}\cos(tb)&\sin(tb)\\-\sin(tb)&\cos(tb)\end{bmatrix}}_{=R_{tb}}$$

R_{tb} → матрица ротације за угао tb

(2) $A = \begin{bmatrix}0&1\\0&0\end{bmatrix}$, $B = \begin{bmatrix}0&0\\-1&0\end{bmatrix}$

Да ли је $e^{tA}\cdot e^{tB} = e^{t(A+B)}$ $(\forall t\in\mathbb{R})$? $(*)$

$$\left.\begin{array}{l} AB = \begin{bmatrix}0&1\\0&0\end{bmatrix}\cdot\begin{bmatrix}0&0\\-1&0\end{bmatrix} = \begin{bmatrix}-1&0\\0&0\end{bmatrix} \\[2ex] BA = \begin{bmatrix}0&0\\-1&0\end{bmatrix}\cdot\begin{bmatrix}0&1\\0&0\end{bmatrix} = \begin{bmatrix}0&0\\0&-1\end{bmatrix}\end{array}\right\} \neq \Rightarrow A \text{ и } B \text{ не комут.}$$

$\left[\ \not\Rightarrow^{(2)}\ \text{не важи } (*)\right]$

ручно испитујемо:

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = \frac{t^0}{0!} A^0 + \frac{t^1 A^1}{1!} = E + tA = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow A^k = A^2, k \geq 2$$

$$e^{tA} \cdot e^{tB} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix} = \begin{bmatrix} 1-t^2 & t \\ -t & 1 \end{bmatrix}$$

$$e^{tB} = \sum_{k=0}^{\infty} \frac{t^k}{k!} B^k = E + tB = \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow B^k = B^2, k \geq 2$$

$$e^{t(A+B)} = e^{t\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

г) $a=0$, $b=1$

$$\begin{bmatrix} 1-t^2 & t \\ -t & 1 \end{bmatrix} \neq \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

(3) Испитати да ли постоји $A \in M_2(\mathbb{R})$ тд:

а) $e^A = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$ б) $e^A = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}$

Γ у $\mathbb{R}$: $e^a = -3$ нема реш.

а) $\underbrace{\det e^A}_{\begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}} = e^{trA}$ (6)

$e^{trA} = \det\left(\begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}\right) = -4$ ↯

б) $\det e^A = \det\left(\begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}\right) = 4 = e^{trA} \Rightarrow trA = \ln 4 \ldots$

(3) $AB = BA \Rightarrow Be^A = e^A B$

$B = A$: $AA = A^2 = AA \overset{(3)}{\Rightarrow} Ae^A = e^A A$

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \cdot \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

$$\begin{bmatrix} -\alpha & -4\beta \\ \boxed{-\gamma} & -4\delta \end{bmatrix} = \begin{bmatrix} -\alpha & -\beta \\ \boxed{-4\gamma} & -4\delta \end{bmatrix} \Rightarrow \left.\begin{array}{l} -4\beta = -\beta \\ -\gamma = -4\gamma \end{array}\right\} \Rightarrow \beta = \gamma = 0$$

$$\Rightarrow A = \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix} \xrightarrow{A^k = \begin{bmatrix} \alpha^k & 0 \\ 0 & \delta^k \end{bmatrix}} e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} \alpha^k & 0 \\ 0 & \delta^k \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{\delta^k}{k!} \end{bmatrix} = \begin{bmatrix} e^\alpha & 0 \\ 0 & e^\delta \end{bmatrix}$$

$$e^A = \begin{bmatrix} e^\alpha & 0 \\ 0 & e^\delta \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \ ↯$$

$$e^\alpha = -1$$
$$e^\delta = -4$$

(4) $\lambda \in \mathbb{C}$ сопств. вредности од A, тогаш је e^λ сопс. вр. од e^A.

$Av = \lambda v \quad , v \neq 0$

? $e^A v' = e^\lambda v'$?

$$A^2 v = A(Av) = A(\lambda v) = \lambda (Av) = \lambda(\lambda v) = \lambda^2 v$$

$\vdots$ (инд.) $A^{k-1} v = \lambda^{k-1} v$

$$A^k v = A(A^{k-1} v) = A(\lambda^{k-1} v) = \lambda^{k-1}(Av) = \lambda^{k-1} \cdot \lambda v = \lambda^k v \Rightarrow \lambda^k \text{ је с.в. од } A^k$$

$$e^A v = \left(\sum_{k=0}^{\infty} \frac{A^k}{k!} \right) \cdot v = \sum_{k=0}^{\infty} \frac{1}{k!} (A^k v) = \sum_{k=0}^{\infty} \frac{\lambda^k v}{k!} = \left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right) \cdot v = e^\lambda \cdot v$$

$\Rightarrow e^\lambda$ сопс. вр. од e^A (за исти сопс. век.)

(5) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Најди $\det(e^{e^A})$.

$$A \in M_3(\mathbb{R}) \Rightarrow e^A \in M_3(\mathbb{R}) \Rightarrow e^{e^A} \in M_3(\mathbb{R})$$

(6) $d = \det\left(e^{\overset{=B}{e^A}}\right) = \det(e^B) = e^{\operatorname{tr} B} = e^{\operatorname{tr}(e^A)}$

$$e^A = ? \qquad A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D} + \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{N}$$

$$DN = ND \nearrow D = E \ \checkmark$$

$$\Downarrow (2)$$

$$e^{tA} = e^{tD} \cdot e^{tN}$$

$$e^{tD} = e^{tE} = \sum_{k=0}^{\infty} \frac{t^k E}{k!} = e^t E$$

$$N^2 = \begin{bmatrix} & & 1 \\ & & \\ 1 & & \end{bmatrix} \cdot \begin{bmatrix} & & 1 \\ & & \\ 1 & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$N^3 = N^2 N = \begin{bmatrix} 1 & & \\ & & \\ & & 1 \end{bmatrix} \cdot \begin{bmatrix} & & 1 \\ & & \\ 1 & & \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = N \Bigg\} \quad \begin{matrix} N^{2k} = N^2 \\ N^{2k+1} = N \end{matrix}$$

$$N^4 = N^3 \cdot N = N \cdot N = N^2$$

$$e^{tN} = \sum_{k=0}^{\infty} \frac{t^k}{k!} N^k = \underset{\substack{\uparrow \\ k=0}}{E} + \sum_{k=1}^{\infty} \frac{t^{2k} \overbrace{N^{2k}}^{= N^2}}{(2k)!} + \sum_{k=0}^{\infty} \frac{t^{2k+1} \overbrace{N^{2k+1}}^{= N}}{(2k+1)!} =$$

$$= E + \sum_{k=1}^{\infty} \frac{t^{2k}}{(2k)!} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 + \operatorname{ch} t - 1 & 0 & \operatorname{sh} t \\ 0 & 1 & 0 \\ \operatorname{sh} t & 0 & 1 + \operatorname{ch} t - 1 \end{bmatrix} = \begin{bmatrix} \operatorname{ch} t & 0 & \operatorname{sh} t \\ 0 & 1 & 0 \\ \operatorname{sh} t & 0 & \operatorname{ch} t \end{bmatrix}$$

$$e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!} \qquad e^t + e^{-t} = 2 \cdot \text{четни}$$

$$e^{-t} = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \qquad e^t - e^{-t} = 2 \cdot \text{нечетни}$$

$$\sum_{k=1}^{\infty} \frac{t^{2k}}{(2k)!} = \underbrace{\frac{1}{2}\left(e^t + e^{-t}\right)}_{\operatorname{ch} t} - 1$$

$$\sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} = \frac{1}{2}\left(e^t - e^{-t}\right)$$

$$\sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} = \underbrace{\frac{1}{2}\left(e^{t} - e^{-t}\right)}_{\operatorname{sh} t}$$

$$t=1: \quad e^{D} = \begin{bmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{bmatrix}$$

$$e^{N} = \begin{bmatrix} \operatorname{ch} 1 & 0 & \operatorname{sh} 1 \\ 0 & 1 & 0 \\ \operatorname{sh} 1 & 0 & \operatorname{ch} 1 \end{bmatrix}$$

$$e^{A} = e^{D} \cdot e^{N} = e \cdot \underbrace{E \cdot e^{N}} = e \cdot e^{N}$$

$$\operatorname{tr}(e^{A}) = e \cdot \operatorname{tr}(e^{N}) = e(\operatorname{ch} 1 + 1 + \operatorname{ch} 1) = e\left(1 + e + \frac{1}{e}\right) \quad \Rightarrow d = e^{e^2+e+1}.$$