Експоненти матрице

$X' = AX$ ЛСДЈКК

$A \in M_n(\mathbb{R})$

$A = [a_{ij}]_{i,j=1}^{n}$

$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $X, X' \in \mathbb{R}^n$

$$\left.\begin{array}{l} x_1' = a_{11}x_1 + \dots + a_{1n}x_n \\ \vdots \\ x_n' = a_{n1}x_1 + \dots + a_{nn}x_n \end{array}\right.$$

ОР: $X(t) = e^{tA} \cdot C$, $C \in \mathbb{R}^n$ $C = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

$\exp : M_n(\mathbb{R}) \to M_n(\mathbb{R})$, $\exp(A) = e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$

Тврђење 52. **(Својства експонента.)**

$\mathbf{0}$ - нула матрица, $\mathrm{Id} = E \in M_n(\mathbb{R})$

(1) $e^{\mathbf{0}} = \mathrm{Id}$;
(2) $AB = BA \Rightarrow e^{A+B} = e^A e^B$;
(3) $AB = BA \Rightarrow Be^A = e^A B$;
(4) $e^A = \lim\limits_{n\to\infty} \left(\mathrm{Id} + \frac{A}{n}\right)^n$;
(5) за $U = \mathbb{R}^n$, тј. $A \in M_n(\mathbb{R})$ важи $\frac{d}{dt}e^{tA} = \boxed{e^{tA}A = Ae^{tA};}$ $(3) \Rightarrow (tA \cdot A = A \cdot tA \Rightarrow e^{tA} \cdot A = A \cdot e^{tA})$
(6) за $U = \mathbb{R}^n$ важи $\det e^A = e^{\operatorname{tr} A}$;
(7) за $U = \mathbb{R}^n$ важи $e^{P^{-1}AP} = P^{-1}e^A P$.

① Решити систем ДЈ $X' = AX$, одређивањем e^{tA} у облику степеног реда, ако је:

а) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ б) $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ в) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ г) $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, $a, b \in \mathbb{R}$

битни

а) $X(t) = \underbrace{e^{tA}}_{?} \cdot c$, $c \in \mathbb{R}^2$

$$e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \; ?$$

$$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$\left[\,0 \quad 1\,\right]\left[\,0 \quad 1\,\right] = \left[\,0 \quad 1\,\right]$

$$A^{3} = A^2 A = \begin{bmatrix}1&2\\0&1\end{bmatrix}\begin{bmatrix}1&1\\0&1\end{bmatrix} = \begin{bmatrix}1&3\\0&1\end{bmatrix}$$

$\vdots$

$$A^{k} = \begin{bmatrix}1&k\\0&1\end{bmatrix} \leftarrow \text{индукција}$$

б) $k=1$ ✓

х) $A^k = \begin{bmatrix}1&k\\0&1\end{bmatrix}$

к) $A^{k+1} = A^k A = \begin{bmatrix}1&k\\0&1\end{bmatrix}\begin{bmatrix}1&1\\0&1\end{bmatrix} = \begin{bmatrix}1&k+1\\0&1\end{bmatrix}$ ✓

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\cdot\begin{bmatrix}1&k\\0&1\end{bmatrix} = \begin{bmatrix}\sum\limits_{k=0}^{\infty}\frac{t^k}{k!} & \sum\limits_{k=0}^{\infty}\frac{t^k}{k!}\cdot k\\ 0 & \sum\limits_{k=0}^{\infty}\frac{t^k}{k!}\end{bmatrix} = \begin{bmatrix}e^t & t\cdot e^t\\ 0 & e^t\end{bmatrix}$$

$$\sum_{k=0}^{\infty}\frac{t^k}{k!} = e^t \quad , \quad \sum_{k=0}^{\infty}\frac{t^k}{k!}\cdot k = \sum_{k=1}^{\infty}\frac{t^k}{(k-1)!} = t\cdot\underbrace{\sum_{k=1}^{\infty}\frac{t^{k-1}}{(k-1)!}} = t\cdot\sum_{k=0}^{\infty}\frac{t^k}{k!} = t\cdot e^t$$

б) домаћи: исто као а)

$$A = \begin{bmatrix}1&1\\0&2\end{bmatrix}$$

идеја: $A = \underbrace{\begin{bmatrix}\ldots\end{bmatrix}}_{D} + \underbrace{\begin{bmatrix}\ldots\end{bmatrix}}_{N}$

$DN = NK$ и лако је наћи e^{tD} и e^{tN}

$\hookrightarrow e^{tA} = e^{t(D+N)} = e^{tD+tN} \overset{(2)}{=} e^{tD}\cdot e^{tN}$

$\underline{tN}\cdot\underline{tD} = t^2\cdot ND = t^2\cdot DN = \underline{tD}\cdot\underline{tN}$

Често: $D = \begin{bmatrix}1&0\\0&2\end{bmatrix}$, $N = \begin{bmatrix}0&1\\0&0\end{bmatrix}$

$$DN = \begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix} = \begin{bmatrix}0&1\\0&0\end{bmatrix} \;\neq\; \Rightarrow \text{не можемо (2)}$$

$$ND = \begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix} = \begin{bmatrix}0&2\\0&0\end{bmatrix}$$

$$ND = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$ $\not\parallel \Rightarrow$ не можемо (2)

Пробамо: $D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ и $N = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ $\to$ свака матрица комутира са $D = E$

$AE = EA = A$

$$\left.\begin{aligned} DN &= EN = N \\ ND &= NE = N \end{aligned}\right\} \Rightarrow \text{можемо (2)}$$

$$\Downarrow$$

$$e^{tA} = \underline{e^{tD}} \cdot \underline{e^{tN}}$$

$$e^{tD} = \sum_{k=0}^{\infty} \frac{t^k D^k}{k!} = \underbrace{\sum_{k=0}^{\infty} \frac{t^k}{k!}}_{=e^t} \cdot E = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} = e^t \cdot E$$

$D^k = E^k = E$

$N^k = ?$ $\quad N^2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = N \quad \Rightarrow N^k = N = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \forall k.$

$$e^{tN} = \sum_{k=0}^{\infty} \frac{t^k N^k}{k!} = e^t \cdot N$$

$$e^{tA} = e^{tD} \cdot e^{tN} = e^t \cdot E \cdot e^t \cdot N = e^{2t} \cdot N = \begin{bmatrix} 0 & e^{2t} \\ 0 & e^{2t} \end{bmatrix}$$

г) $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \longleftrightarrow a + ib$ $\qquad e^{tA} \longleftrightarrow e^{t(a+ib)} = e^{ta} \cdot (\cos(bt) + i\sin(bt))$

$$e^{tA} = e^{ta} \cdot \underbrace{\begin{bmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{bmatrix}}_{R_{tb}\text{ – матрица ротације}}$$

даље: $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \dots$

(2) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$

(*) Да ли је $e^{tA}\cdot e^{tB} = e^{t(A+B)}$? $(\forall t\in\mathbb{R})$

$$AB = \begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}0&0\\-1&0\end{bmatrix} = \begin{bmatrix}-1&0\\0&0\end{bmatrix}$$

$$BA = \begin{bmatrix}0&0\\-1&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix} = \begin{bmatrix}0&0\\0&-1\end{bmatrix}$$

$\neq \Rightarrow$ не комутирају (2) [(*) не важи $(\forall A)$]

ручно:

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\cdot A^k = \frac{t^0}{0!}\cdot A^0 + \frac{t^1}{1!}A^1 = E + tA = \begin{bmatrix}1&t\\0&1\end{bmatrix}$$

$$A^2 = \begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix} \Rightarrow A^k = A^2, \; k\geq 2$$

$$e^{tB} = \sum_{k=0}^{\infty}\frac{t^k}{k!}B^k = E + tB = \begin{bmatrix}1&0\\-t&1\end{bmatrix}$$

$$B^2 = \begin{bmatrix}0&0\\-1&0\end{bmatrix}\begin{bmatrix}0&0\\-1&0\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}, \; B^k = B^2, \; k\geq 2$$

$$e^{tA}\cdot e^{tB} = \begin{bmatrix}1&t\\0&1\end{bmatrix}\begin{bmatrix}1&0\\-t&1\end{bmatrix} = \begin{bmatrix}1-t^2&t\\-t&1\end{bmatrix}$$

$$e^{t(A+B)} = \exp\left(t\cdot\begin{bmatrix}0&1\\-1&0\end{bmatrix}\right) = \begin{bmatrix}\cos t&\sin t\\-\sin t&\cos t\end{bmatrix}$$

1г)
$a=0$
$b=1$

$$\begin{bmatrix}1-t^2&t\\-t&1\end{bmatrix} \neq \begin{bmatrix}\cos t&\sin t\\-\sin t&\cos t\end{bmatrix}$$

(3) Испитати да ли постоји $A\in M_2(\mathbb{R})$ тд:

а) $e^A = \begin{bmatrix}1&0\\0&-4\end{bmatrix}$ б) $e^A = \begin{bmatrix}-1&0\\0&-4\end{bmatrix}$

у $\mathbb{R}$: $e^a = -1$ нема реш.

а) $\det(e^A) = e^{trA}$ (6)

$\Downarrow$

$\det(e^A) > 0$

$\det(e^A) = \det\left(\begin{bmatrix}1&0\\0&-4\end{bmatrix}\right) = -4 < 0$ ↯

б) $\det(e^A) = \det\left(\begin{bmatrix}-1&0\\0&-4\end{bmatrix}\right) = 4 > 0$

$\Vert$

e^{trA} $\Rightarrow trA = \ln 4 \ldots$

(3) $\Rightarrow$ $(AB = BA \Rightarrow Be^A = e^A B)$

$$A\cdot A = A^2 = A\cdot A \overset{(3)}{\Longrightarrow} Ae^A = e^A A \qquad\qquad A=\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\cdot\begin{bmatrix}-1 & 0\\ 0 & -4\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -4\end{bmatrix}\cdot\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}$$

$$\begin{bmatrix}-\alpha & \boxed{-4\beta}\\ \boxed{-\gamma} & -4\delta\end{bmatrix} = \begin{bmatrix}-\alpha & \boxed{-\beta}\\ \boxed{-4\gamma} & -4\delta\end{bmatrix} \Longrightarrow \left.\begin{matrix}-4\beta=-\beta\\ -\gamma=-4\gamma\end{matrix}\right\} \beta=\gamma=0$$

индукција

$$\Rightarrow A=\begin{bmatrix}\alpha & 0\\ 0 & \delta\end{bmatrix} \xrightarrow{A^k=\begin{bmatrix}\alpha^k & 0\\ 0 & \delta^k\end{bmatrix}} e^A=\sum_{k=0}^{\infty}\frac{A^k}{k!}=\sum_{k=0}^{\infty}\frac{\begin{bmatrix}\alpha^k & 0\\ 0 & \delta^k\end{bmatrix}}{k!}=\begin{bmatrix}\sum\limits_{k=0}^{\infty}\frac{\alpha^k}{k!} & 0\\ 0 & \sum\limits_{k=0}^{\infty}\frac{\delta^k}{k!}\end{bmatrix}=\begin{bmatrix}e^\alpha & 0\\ 0 & e^\delta\end{bmatrix}$$

$$e^A=\begin{bmatrix}e^\alpha & 0\\ 0 & e^\delta\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & -4\end{bmatrix} \Rightarrow \begin{matrix}e^\alpha=-1\\ e^\delta=-4\end{matrix}\ \lightning$$

(4) $\lambda\in\mathbb{C}$ сопст. вредност од A, онда је e^λ сопс. вр. од e^A.

$$A\,|\; Av=\lambda v \quad , v\neq 0 \qquad\qquad e^A v' = e^\lambda v'\,?$$

$$A^2v = A(\lambda v)=\lambda(Av)=\lambda(\lambda v)=\lambda^2 v$$

$\vdots$ инд.

$$A^k v=\lambda^k v \Rightarrow \lambda^k \text{ сопс вр. од } A^k$$

$$A^k v = A(A^{k-1}v) = A(\lambda^{k-1}v) = \lambda^{k-1}(Av) = \lambda^{k-1}(\lambda v) = \lambda^k v$$

$$e^A v = \left(\sum_{k=0}^{\infty}\frac{A^k}{k!}\right)v = \sum_{k=0}^{\infty}\frac{A^k v}{k!} = \sum_{k=0}^{\infty}\frac{\lambda^k v}{k!} = \underbrace{\left(\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\right)}_{e^\lambda} v = e^\lambda v$$

$\Rightarrow e^\lambda$ је с. в. од e^A (за исти сопс. вектор v)

(5) $A=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}$. Најти $d=\det\left(e^{e^A}\right)$.

$$A\in M_3(\mathbb{R}) \Rightarrow e^A\in M_3(\mathbb{R}) \Rightarrow e^{e^A}\in M_3(\mathbb{R})$$

$(6) \Rightarrow \det(e^{\square}) = e^{\operatorname{tr}\square}$

$\square = e^{A}$

$d = \det(e^{e^{A}}) = e^{\operatorname{tr}(e^{A})}$

$e^{A} = ?$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E} + \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{B}$$

$EB = B = BE$

$\Downarrow$

$e^{A} = e^{E+B} = e^{E} \cdot e^{B}$

$$e^{E} = \sum_{k=0}^{\infty} \frac{E^k}{k!} = \underbrace{\sum_{k=0}^{\infty} E \cdot \frac{1}{k!}}_{=e} = eE = \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}$$

$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow e^{E} = \begin{bmatrix} e^1 & 0 \\ 0 & e^1 \end{bmatrix}$

$+ = 0$

$$B^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B^3 = B^2 B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = B$$

$$\begin{cases} B^{2k+1} = B, & k \in \mathbb{N}_0 \\ B^{2k} = B^2, & k \in \mathbb{N} \end{cases}$$

$B^4 = B^3 \cdot B = B \cdot B = B^2$

$$e^{tB} = \sum_{k=0}^{\infty} \frac{t^k B^k}{k!} = \underset{k=0}{E} + \sum_{k=1}^{\infty} \frac{t^{2k} B^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{t^{2k+1} B^{2k+1}}{(2k+1)!} =$$

$B^{2k} = B^2$, $B^{2k+1} = B$

$$= E + B^2 \cdot \sum_{k=1}^{\infty} \frac{t^{2k}}{(2k)!} + B \cdot \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} = \begin{bmatrix} \operatorname{ch} t & 0 & \operatorname{sh} t \\ 0 & 1 & 0 \\ \operatorname{sh} t & 0 & \operatorname{ch} t \end{bmatrix}$$

$$e^{t} = \sum_{k=0}^{\infty} \frac{t^k}{k!}$$

$$e^{-t} = \sum_{k=0}^{\infty} \frac{(-1)^k t^k}{k!}$$

$e^{t} + e^{-t} = 2 \cdot$ парни

$e^{t} - e^{-t} = 2 \cdot$ непарни

$$e^{-t} = \sum_{k=0} \frac{(-1)^k t^k}{k!}$$

$e^t - e^{-t}$ $= 2 \cdot$ нечетни /

$$\sum_{k=1}^{\infty} \frac{t^{2k}}{(2k)!} = \underset{-(k=0)}{-1} + \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} = -1 + \underbrace{\frac{e^t + e^{-t}}{2}}_{\operatorname{ch} t} = \underline{\operatorname{ch} t - 1}$$

$$\sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} = \underbrace{\frac{e^t - e^{-t}}{2}}_{\operatorname{sh} t} = \operatorname{sh} t$$

$$\operatorname{ch} 1 = \frac{e + e^{-1}}{2}$$

$$\operatorname{sh} 1 = \frac{e - e^{-1}}{2}$$

$t = 1$: $e^E = e \cdot E$

$$e^B = \begin{bmatrix} \operatorname{ch} 1 & 0 & \operatorname{sh} 1 \\ 0 & 1 & 0 \\ \operatorname{sh} 1 & 0 & \operatorname{ch} 1 \end{bmatrix}$$

$$e^A = e^E \cdot e^B = e \cdot E \cdot e^B \Rightarrow tr(e^A) = e \cdot tr(e^B) = e(2\operatorname{ch} 1 + 1) = e + e^2 + 1$$

$$\Rightarrow d = e^{e^2 + e + 1}$$