7 ДЈ са тоталним диференцијалом

ТОТ. Д.

$$M(t,x)dt + N(t,x)dx = 0$$

$$x' = \frac{f(t,x)}{g(t,x)} \iff \frac{dx}{dt} = \frac{f(t,x)}{g(t,x)}$$

$$\iff f(t,x)dt = g(t,x)dx$$

$$\iff f(t,x)dt - g(t,x)dx = 0$$

у диференцијалном облику

$\boxed{\exists F(t,x)}$

$$dF(t,x) = M(t,x)dt + N(t,x)dx$$

$$= \frac{\partial F}{\partial t}(t,x)dt + \frac{\partial F}{\partial x}(t,x)dx$$

$$\left.\begin{array}{l} M = F'_t \\ N = F'_x \end{array}\right\} \Rightarrow M'_x = N'_t$$

$$\text{ТОТ. Д.} \Rightarrow M'_x = N'_t$$

$\boxed{\Leftarrow}$ (до нас зависи) ← важи, али кад радимо у просто-повезаној (пп) области

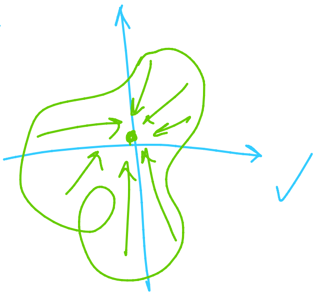
↑ ту посматрамо ДЈ (нпр. ту је деф.)

Како да знамо да имамо ДЈ са ТОТ. Д?

просто-повезане (пп) области

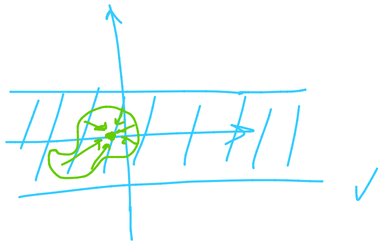
D је пп $\iff$ сваку непрекидну затворену криву у D можемо скупити у тачку кроз D

пр. 1) $\mathbb{R}^2$



✓

2) траку



✓

„пп (у $\mathbb{R}^2$) акко НЕМА РУПЕ"

X

↓

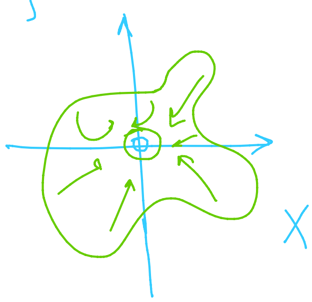
у вишим дим. НЕ!

→ $\mathbb{R}^3 \setminus \{(0,0,0)\}$ јесте пп

→ $\mathbb{R}^3 \setminus \ell$ није пп

↑ права

3) $\mathbb{R}^2 \setminus \{(0,0)\}$



X

$$0 = Mdt + Ndx = dF$$

$\boxed{\text{OP: } F(t,x) = c, \ c \in \mathbb{R}}$ ← имплицитно задато

(1) a) $2t(1+\sqrt{t^2-x})dt - \sqrt{t^2-x}\,dx = 0$

б) $(1+x^2\sin 2t)\,dt - x^2\cos^2 t\,dx = 0$

в) $(t\ln x + tx)\,dt + (x\ln t + tx)\,dx = 0$

г) $(tx^2+3t^2x)\,dt + (t^3+t^2x)\,dx = 0$

д) $M(t,x) = 2t(1+\sqrt{t^2-x}) = 2t+2t\sqrt{t^2-x}$

$N(t,x) = -\sqrt{t^2-x}$

$$M'_x = 2t\cdot\frac{1}{2\sqrt{t^2-x^2}}\cdot(-1)$$

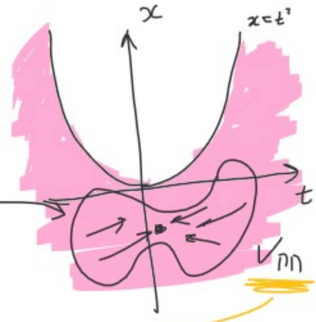
$$\parallel \quad \Rightarrow M'_x = N'_t$$

$$N'_t = -\frac{1}{2\sqrt{t^2-x}}\cdot 2t$$

област?

$t^2-x\geq 0$

$x\leq t^2$



тот. д.

$\exists F=?\ ,\ dF = M\,dt+N\,dx$

$$F'_t = M = 2t+2t\sqrt{t^2-x}$$

$$F'_x = N = -\sqrt{t^2-x} \longrightarrow F(t,x) = \int -\sqrt{t^2-x}\,dx = \frac{2}{3}(t^2-x)^{3/2} + \frac{\varphi(t)}{\cancel{(t\,c?)}}$$

$$\frac{2}{3}\cdot\frac{3}{2}\cdot(t^2-x)^{1/2}\cdot 2t + \varphi'(t) = 2t+2t\sqrt{t^2-x}$$

$\Rightarrow \varphi'(t) = 2t$ ← ако не добијемо само по t, онда је грешка

$\varphi(t) = t^2$

$$\boxed{\text{ОР: } \frac{2}{3}(t^2-x)^{3/2}+t^2 = c\in\mathbb{R}}$$

8 Интеграциони фактор

$$M(t,x)\,dt+N(t,x)\,dx = 0$$

$M'_x \neq N'_t \Rightarrow$ није тот. д.

Идеја: $\mu(t,x)\cdot\big(M(t,x)\,dt + N(t,x)\,dx\big) = 0$

инт. фактор

тот. д.

$\mu : A_{\subseteq\mathbb{R}} \to \mathbb{R}$

$w : D_{\subseteq\mathbb{R}^2} \to \mathbb{R}$

$\mu(t,x)$ често у облику $\mu = \mu(w) = \mu(w(t,x))$

Предавања: $$\frac{\mu'(w)}{\mu(w)} = \frac{N'_t - M'_x}{w'_x M - w'_t\cdot N}$$

$$\frac{d\mu}{\mu} = \frac{N'_t - M'_x}{w'_x \cdot M - w'_t \cdot N}\, dw \quad \Big/ \int$$

→ зависи само од w

Циљ: наћи w !

Чести облици за w:

- $w(t,x) = at + bx$ $\quad (w = x,\ w = t, \ldots)$
- $w(t,x) = a\ln|t| + b\ln|x|$
- $w(t,x) = f(t)\cdot g(x)$
- $w(t,x) = f(t) + g(x)$

[проверити $\frac{1}{\mu} = 0$ за решење које је можда изгубљено у рачуну]

(2) а) $(q(t) - p(t)x)dt - dx = 0$, $p, q: \mathbb{R} \to \mathbb{R}$ непр. и познате ← упоредити са пр. 1) $(w = t)$

б) $2tx\ln x\, dt + (t^2 + x^2\sqrt{x^2+1})\, dx = 0$

в) $x(2 - 3tx^2)\, dt - t(1 + tx^2)\, dx = 0$, на области $G = \{x > 0, t > 0\}$, наћи решење које пролази кроз $(2,1)$

г) $(\sqrt{t^2 - x} + 2t)dt - dx = 0$, μ тражити у облику $\mu(t^2 - x)$ $\quad (w = t^2 - x$

д) $(t+2)\sin x\, dt + 2t\cos x\, dx = 0$ $\quad (w = x)$

$$\left(x' = \frac{dx}{dt} = -\frac{t+2}{2t} \cdot \operatorname{tg} x \right) \quad \text{РП}$$

б) $M(t,x) = 2tx\ln x$

$N(t,x) = t^2 + x^2\sqrt{x^2+1}$

$$\left.\begin{aligned} M'_x &= 2t\,(x\ln x)'_x = 2t(1 + \ln x) \\ N'_t &= 2t \end{aligned}\right\} \quad \begin{aligned} \mu &= ? \\ w &= ? \end{aligned}$$

$$\frac{d\mu}{\mu} = \frac{N'_t - M'_x}{w'_x \cdot M - w'_t \cdot N}\, dw = \frac{-2t\cdot\ln x}{w'_x\cdot(2tx\ln x) - w'_t\cdot(t^2 + x^2\sqrt{x^2+1})}\, dw = \frac{\cancel{-2t\ln x}}{1\cdot(\cancel{2t}x\,\cancel{\ln x}) - 0\cdot(\ldots)}\, dw = -\frac{dw}{x} = -\frac{dw}{w}$$

✓ добијамо

узимамо: $\left.\begin{aligned} w'_t &= 0 \\ w'_x &= ? = 1 \end{aligned}\right\}$ $w = x$ → најједноставније

$$\frac{d\mu}{\mu} = -\frac{dw}{w} \quad \Big/ \int$$

$\ln|\mu| = -\ln|w| \ \cancel{(+c)} \Rightarrow \mu = 1 \ldots$

$$\mu = -\overline{w} / J$$

$$\ln|\mu| = -\ln|w| \cancel{(+c)} \rightsquigarrow \mu = \frac{1}{w}$$

$$\mu(t,x) = \mu(w(t,x)) = \frac{1}{w(t,x)} = \frac{1}{x} \checkmark$$

$\frac{1}{\mu} = 0 \Leftrightarrow$

$x = 0$? ×

$(\ln \underline{x}, x>0)$

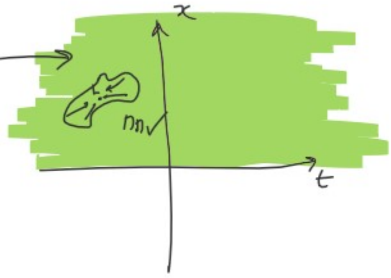
$\mu M dt + \mu N dx = 0 \rightsquigarrow$ тот. д.

$$\underbrace{2t\ln x}_{= F_t'}\, dt + \underbrace{\left(\frac{t^2}{x} + x\sqrt{x^2+1}\right)}_{= F_x'} dx = 0$$

$F = ?$

области?

$\ln x \rightsquigarrow x>0$



$$F_t' = 2t\ln x \rightsquigarrow F(t,x) = t^2\ln x + \varphi(x)$$

$$\frac{t^2}{\cancel{x}} + x\sqrt{x^2+1} = \frac{t^2}{\cancel{x}} + \varphi'(x) \therefore \varphi \ldots$$

ОР:

$$\boxed{F(t,x) = t^2\ln x + \frac{1}{3}(x^2+1)^{3/2} = C,\ C\in\mathbb{R}}$$

В) $M(t,x) = 2x - 3tx^3$

$N(t,x) = -t - t^2x^2$

$M_x' = 2 - 9tx^2$

$N_t' = -1 - 2tx^2$

$$\frac{d\mu}{\mu} = \frac{N_t' - M_x'}{w_x'\cdot M - w_t'\cdot N} dw = \frac{-3+7tx^2}{w_x'\cdot(2x-3tx^3) + w_t'\cdot(t+t^2x^2)} dw = \frac{-3+7tx^2}{(2b-3btx^2)+(a+atx^2)} dw = \frac{-3+7tx^2}{(2b+a)+tx^2(-3b+a)} dw$$

x, x, t, t ← сокращать

const, tx^2, const, tx^2

? $w_x' = \frac{1}{x}$?

$w_t' = \frac{1}{t}$

$w(t,x) = a\ln \underline{t} + b\ln \underline{x}$ ← пробуем

$(t>0,\ x>0)$

$$w_x' = \frac{b}{x},\quad w_t' = \frac{a}{t}$$

хотим:

$$\left.\begin{aligned} -3 &= 2b+a \\ 7 &= -3b+a \end{aligned}\right\} - \Rightarrow -10 = 5b \Rightarrow b = -2 \Rightarrow a = 1$$

$$w = \ln t - 2\ln x$$

$$d\mu \quad \Rightarrow \mu = e^w \Rightarrow \mu(t,x) = e^{\ln t - 2\ln x} = \frac{t}{x^2}$$

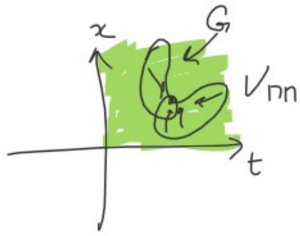
$w = \ln t - 2\ln x$

$$\frac{d\mu}{\mu} = dw \Rightarrow \mu = e^{w} \Rightarrow \mu(t,x) = e^{\ln t - 2\ln x} = \frac{t}{x^2}$$

$\cdot \mu$:

$$\left(2\frac{t}{x} - 3t^2x\right)dt - \left(\frac{t^2}{x^2} + t^3\right)dx = 0$$

область?



$\Rightarrow$ тот. д. $\Rightarrow$?F

$$\left.\begin{aligned} F'_t &= 2\frac{t}{x} - 3t^2x \\ F'_x &= -\left(\frac{t^2}{x^2} + t^3\right) \end{aligned}\right\} \ldots$$

ОР: $\underbrace{F(t,x) = \frac{t^2}{x} - t^3x = C,\ C \in \mathbb{R}}$

ПР: $(2,1) \longrightarrow C = ?$

$t=2$, $x=1$

$$\frac{2^2}{1} - 2^3 \cdot 1 = C = 4 - 8 = -4$$

ПР: $\boxed{\frac{t^2}{x} - t^3x = -4}$