5 Бернулијева ДЈ

$$x' + p(t)x = q(t)x^{\alpha}, \quad \alpha \in \mathbb{R} \qquad p, q : (a,b) \to \mathbb{R} \text{ непр.}$$

СМЕНА: $y(t) = x(t)^{1-\alpha}$

$$\left.\begin{aligned} y(t) &= x(t)^{1-\alpha} \\ y' &= (x^{1-\alpha})' = (1-\alpha)\cdot x^{-\alpha}\cdot x' \end{aligned}\right\} \rightsquigarrow \text{лин}$$

1) а) $x' = \frac{x}{t} - x^2$

б) $x' + \frac{x}{t} = x^2 \frac{\ln t}{t}$

в) $tx' - 2t\sqrt{x} = 4x$

б) $x' + \frac{x}{t} = x^2 \frac{\ln t}{t} \;/ : x^2$ $\quad \alpha = 2$

$t > 0$

$y = x^{1-2} = x^{-1} = \frac{1}{x} \Rightarrow x = \frac{1}{y}$

$y' = -\frac{1}{x^2}\cdot x' = -x^{-2}x'$

$$\underbrace{x'x^{-2}}_{-y'} + \frac{\overbrace{x^{-1}}^{y}}{t} = \frac{\ln t}{t}$$

$$-y' + \frac{y}{t} = \frac{\ln t}{t} \;/\cdot(-1)$$

$$y' - \frac{1}{t}y = -\frac{\ln t}{t} \longrightarrow \text{лин.}$$

$p(t) = -\frac{1}{t}$

$q(t) = -\frac{\ln t}{t}$

$$y(t) = e^{-\int p\,dt}\left(c + \int q \cdot e^{\int p\,dt}\,dt\right)$$

$$\int p\,dt = -\int \frac{dt}{t} = -\ln t \qquad e^{-\ln t} = (e^{\ln t})^{-1} = t^{-1}$$

$$\int q\cdot e^{\int p\,dt}\,dt = -\int \frac{\ln t}{t}\cdot t^{-1}\,dt = -\int \frac{\ln t}{t^2}\,dt$$

$$= -\left(-\frac{1}{t}\ln t - \int\left(-\frac{1}{t}\right)\cdot\frac{1}{t}\,dt\right)$$

$$= \frac{\ln t}{t} - \int\frac{dt}{t^2} = \frac{\ln t}{t} + \frac{1}{t}$$

$$\left(\frac{\ln t + 1}{t}\right)' = \frac{\frac{1}{t}\cdot t - 1\cdot(\ln t + 1)}{t^2} = \frac{-\ln t}{t^2} \checkmark$$

$$y(t) = t\cdot\left(c + \frac{\ln t + 1}{t^2}\right)$$

$$\boxed{x(t) = \frac{1}{t\cdot\left(c + \frac{\ln t + 1}{t^2}\right)}, \quad c \in \mathbb{R}}$$

в) $tx' - 2t\sqrt{x} = 4x \;/ : t$

$$x' - 2\sqrt{x} = \frac{4x}{t}$$

$$\left.x' - \frac{1}{t}\cdot 4x = 2\underbrace{\sqrt{x}}_{x^{\alpha}}\right\} \alpha = \frac{1}{2} \quad \ldots$$

6 Рикатијева ДЈ

$$x' = p(t)x^2 + q(t)x + r(t) \qquad p, q, r : (a,b) \to \mathbb{R} \text{ непр.}$$

СМЕНА: $x_p(t)$ – једно партикуларно решење

$$x(t) = x_p(t) + \frac{1}{y(t)} \qquad x(t) \rightsquigarrow y(t)$$

$\rightsquigarrow$ лин

$$x(t) = x_p(t) + \frac{1}{y(t)} \rightsquigarrow \text{ЛНЧ} \qquad x(t) \rightsquigarrow y(t)$$

(2) а) $t(2t-1)x' + x^2 - (4t+1)x + 4t = 0$

б) $x' = \frac{2\sin t - x^2 \sin t \cos^2 t}{\cos^2 t}$

в) $x' + x^2 + \frac{4x}{t} + \frac{2}{t^2} = 0$

а) $x_p = ?$

$$\underbrace{t(2t-1)}_{2\text{ст.}}\underbrace{x'}_{0} + \underbrace{x^2}_{2} - \underbrace{(4t+1)}_{1\text{ст.}}x + \underbrace{4t}_{1} = 0 \quad \Big\} \quad x_p = at + b, \quad a, b \in \mathbb{R}$$

0 ст.

$x_p' = a$

$t(2t-1)\cdot a + (at+b)^2 - (4t+1)(at+b) + 4t = 0$

$2at^2 - at + a^2t^2 + 2abt + b^2 - 4at^2 - 4bt - at - b + t = 0$

$$t^2\underbrace{(2a + a^2 - 4a)} + t(-a + 2ab - 4b - a + 4) + \underbrace{(b^2 - b)} = 0$$

$a^2 - 2a = 0, \quad a(a-2) = 0$

$b^2 - b = 0, \quad b(b-1) = 0$

$=0\,?$

$(a, b) \in \{\cancel{(0,0)}, (0,1), (2,0), (2,1)\}$

$\times \quad \checkmark \quad \checkmark \quad \checkmark$

$x_p(t) = 2t$

$x(t) \rightsquigarrow y(t)$

$$x(t) = x_p(t) + \frac{1}{y(t)}$$

$$x = 2t + \frac{1}{y} \Rightarrow x' = 2 + \left(-\frac{1}{y^2}\right)\cdot y' = 2 - \frac{y'}{y^2}$$

$$t(2t-1)x' + x^2 - (4t+1)x + 4t = 0 \longrightarrow t(2t-1)\cdot\left(2 - \frac{y'}{y^2}\right) + \left(2t + \frac{1}{y}\right)^2 - (4t+1)\left(2t + \frac{1}{y}\right) + 4t = 0 \,/\cdot y^2$$

$$t(2t-1)(2y^2 - y') + (2ty+1)^2 - (4t+1)(2ty^2 + y) + 4ty^2 = 0$$

$$4t^2y^2 - 2t^2y' - 2ty^2 + ty' + 4t^2y^2 + 4ty + 1 - 8t^2y^2 - 4ty - 2ty^2 - y + 4ty^2 = 0$$

$$-2t^2y' + ty' + 1 - y = 0$$

$$y'(t - 2t^2) - y + 1 = 0$$

$$y' - \frac{1}{t - 2t^2}\,y = -\frac{1}{t - 2t^2} \quad \therefore \nearrow \text{ЛНЧ} \rightsquigarrow y(t) \rightsquigarrow x(t)$$

б) $x' = \dfrac{2\sin t - x^2 \sin t \cos^2 t}{\cos^2 t}$

$x' = 2\dfrac{\sin t}{\cos^2 t} - \boxed{x^2 \cdot \sin t} \longrightarrow \dfrac{\sin t}{\cos^2 t}\cdot a$

$\pm \sin = (\cos t)'$

$\boxed{x_p(t) = \dfrac{a}{\cos t}}$, $x_p' = -\dfrac{a\cdot(-\sin t)}{\cos^2 t} = a\dfrac{\sin t}{\cos^2 t}$

в) $x' + x^2 + \dfrac{4x}{t} + \dfrac{2}{t^2} = 0$

$\boxed{x_p = \dfrac{a}{t}}$

$x_p^2 = \dfrac{a^2}{t^2}$

$\dfrac{4x_p}{t} = \dfrac{4a}{t^2}$

$\dfrac{2}{t^2}$

Смене

1 $2t x''(t) + (1+\sqrt{t})x'(t) - x(t) = t + \sin(\sqrt{t})$, $t>0$

$u = \sqrt{t}$

Своди на: $x'' + x' - 2x = 2u^2 + 2\sin u$

$x(t) \rightsquigarrow x(u)$

$\dfrac{dx}{dt} \to \dfrac{dx}{du}$

$\dfrac{dx}{du} = \dfrac{dx}{dt}\cdot\dfrac{dt}{du} = \dfrac{dx}{dt}\cdot 2u \longrightarrow \dfrac{dx}{dt} = \dfrac{1}{2u}\cdot\dfrac{dx}{du}$

$u = \sqrt{t} \Rightarrow t = u^2$

$\dfrac{du}{dt} = \dfrac{1}{2\sqrt{t}} = \dfrac{1}{2u}$

$\dfrac{d^2x}{dt^2} = \dfrac{d}{dt}\left(\dfrac{dx}{dt}\right) = \dfrac{d}{dt}\left(\dfrac{1}{2u}\cdot\dfrac{dx}{du}\right) = \dfrac{d}{dt}\left(\dfrac{1}{2u}\right)\cdot\dfrac{dx}{du} + \dfrac{1}{2u}\cdot\dfrac{d}{dt}\left(\dfrac{dx}{du}\right)$

$= -\dfrac{1}{2u^2}\cdot\dfrac{du}{dt}\cdot\dfrac{dx}{du} + \dfrac{1}{2u}\cdot\dfrac{du}{dt}\cdot\dfrac{d}{du}\left(\dfrac{dx}{du}\right) =$

$\dfrac{d}{du}\left(\dfrac{1}{2u}\right)$

$= -\dfrac{1}{4u^3}\cdot\boxed{\dfrac{dx}{du}} + \dfrac{1}{4u^2}\cdot\boxed{\dfrac{d^2x}{du^2}}$ $\quad = x_u' \quad = x_{uu}''$

$x_t' = x_u' \cdot u_t'$

$x_t'' = (x_u' \cdot u_t')_t' =$

$= x_{uu}'' \cdot u_t' \cdot u_t' + x_u' \cdot u_{tt}''$

$u_{tt}'' = \left(\dfrac{1}{2\sqrt{t}}\right)_t' = \dfrac{1}{2}\left(-\dfrac{1}{2}\right)t^{-3/2}$

$= -\dfrac{1}{4u^3}$

$2t x''(t) + (1+\sqrt{t})x'(t) - x(t) = t + \sin(\sqrt{t})$

$2u^2\cdot\left(-\dfrac{1}{4u^3}\cdot\dfrac{dx}{du} + \dfrac{1}{4u^2}\cdot\dfrac{d^2x}{du^2}\right) + (1+u)\cdot\dfrac{1}{2u}\cdot\dfrac{dx}{du} - x = u^2 + \sin u$

$\boxed{-\dfrac{1}{2u}\cdot\dfrac{dx}{du}} + \dfrac{1}{2}\cdot\dfrac{d^2x}{du^2} \boxed{+\dfrac{1}{2u}\cdot\dfrac{dx}{du}} + \dfrac{1}{2}\dfrac{dx}{du} - x = u^2 + \sin u \,/\cdot 2$

$\dfrac{d^2x}{du^2} + \dfrac{dx}{du} - 2x = 2u^2 + 2\sin u$ ✓ (знаћемо да је решимо касније)

2 $x(t) \rightsquigarrow y(t)$

а) $t x^2 x' + x^3 = t\cos t \longrightarrow y(t) = x(t)^3 \to y' = 3\cdot x^2 \cdot x' \longrightarrow t\cdot\dfrac{y'}{3} + y = t\cos t$ лин.

б) $x'\cos x = \dfrac{\sin x}{t} - \sin^2 x \longrightarrow y(t) = \sin x(t) \to y' = \cos x \cdot x' \longrightarrow y' = \dfrac{y}{t} - y^2$ $\alpha = 2$ БИК. (БЕР) в) $x' \operatorname{tg} x + 4t^3 \cos^3 x = 2t \longrightarrow x' \cdot \frac{\sin x}{\cos x} + 4t^3 \cos^3 x = 2t \to y = \cos x \to y' = -\sin x \cdot x' \to \frac{-y'}{y} + 4t^3 y^3 = 2t$

г) $t e^x x' - 2t e^{x/2} = 4e^x$

$e^x = y \to y' = e^x \cdot x' \to t y' - 2t\sqrt{y} = 4y$ (БЕР) $\alpha = 2$

$y' - 4t^3 y^4 = -2ty$ $\alpha = 4$ (БЕР)

3 $\quad \frac{dx}{dt} = \frac{x((\ln x)^2 + t)}{2t^{3/2}}, \quad t > 0, \; x > 0$

$x(t) \rightsquigarrow y(u)$

$\frac{dx}{dt} \to \frac{dy}{du}$

Решити уводећи: $u = \sqrt{t} \Rightarrow t = u^2$

$y = \ln x \Rightarrow x = e^y$

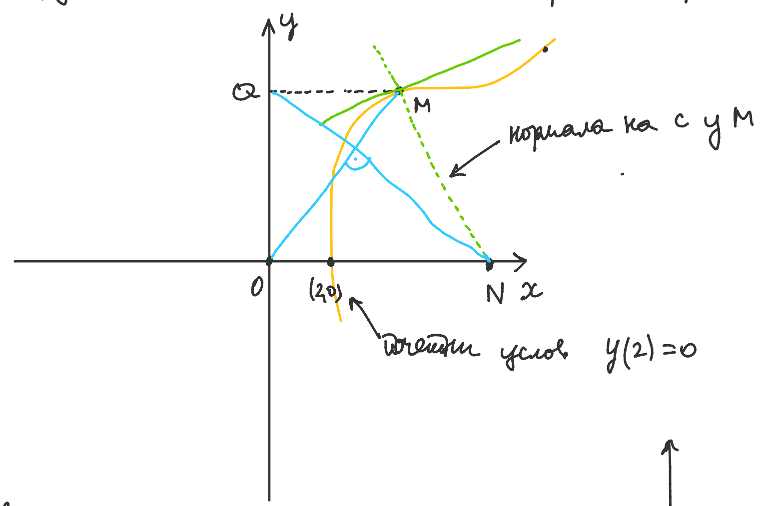
$$\frac{dy}{du} = \underbrace{\boxed{\frac{dy}{dx}}}_{\frac{1}{x} = e^{-y}} \cdot \frac{dx}{dt} \cdot \underbrace{\boxed{\frac{dt}{du}}}_{2u} \Rightarrow \frac{dx}{dt} \cdot \frac{2u}{e^y} = \frac{dy}{du} \Rightarrow \frac{dx}{dt} = \frac{dy}{du} \cdot \frac{e^y}{2u}$$

$$\frac{dy}{du} \cdot \frac{\cancel{e^y}}{\cancel{2u}} = \frac{\cancel{e^y} \cdot (y^2 + u^2)}{\cancel{2} \cdot u^{\cancel{3}\,2}} \Rightarrow \frac{dy}{du} = \frac{y^2 + u^2}{u^2} = \left(\frac{y}{u}\right)^2 + 1 \quad \text{(хом.)}$$

$y(u) \rightsquigarrow x(t) = e^{y(t)} = e^{y(u^2)}$

1 (свака) Тачка M криве c се пројектује на y-осу у Q, а нормала на c у M сече x-осу у N. Нека је O коорд. почетак. Ако $QN \perp OM$ и крива c пролази кроз $(2,0)$, одредити c.

c: $y(x) = ?$



Где је овде диференцијална једначина?