3 - петак 30.10.2020.

5 Бернулијева ДЈ

$x' + p(t)x = q(t)x^\alpha$, $p,q:(a,b)\to\mathbb{R}$ непр. , $\alpha\in\mathbb{R}$

$\alpha=0$ лин

$\alpha=1$ РП

СМЕНА: $y(t)=x(t)^{1-\alpha}$

$y'=(1-\alpha)\cdot x^{-\alpha}\cdot x'$ ⇝ лин.

(1) а) $x'=\frac{x}{t}-x^2$

б) $x'+\frac{x}{t}=x^2\frac{\ln t}{t}$

в) $tx'-2t\sqrt{x}=4x$

в) $x'-2\sqrt{x}=\frac{4}{t}\cdot x \to x'\boxed{-\frac{4}{t}}\cdot x = 2\sqrt{x}$

$p(t)$, $q(t)$, $\sqrt{x}=x^{1/2}$, $\alpha=\frac{1}{2}$

$y=x^{1-\alpha}=x^{1-\frac{1}{2}}=x^{1/2}$

$\Downarrow$

$x=y^2$

$/:\sqrt{x}$

$x'\cdot x^{-1/2}-\frac{4}{t}\cdot x^{1/2}=2$

$y'=\frac{1}{2\sqrt{x}}\cdot x'=\frac{1}{2}\cdot x'\cdot x^{-1/2}$

$2y'-\frac{4}{t}\cdot y=2$

$y'-\frac{2}{t}y=1$

$p(t)=-\frac{2}{t}$

$q(t)=1$

$\int p\,dt=\int -\frac{2}{t}\,dt=-2\ln|t|\ (+c)$

$\int 1\cdot e^{-2\ln|t|}\,dt=\int\frac{1}{t^2}\,dt=-\frac{1}{t}\ (+c)$

$e^{-2\ln|t|}=\left(e^{\ln|t|}\right)^{-2}=|t|^{-2}=\frac{1}{t^2}$

$y=e^{-\int p\,dt}\left(c+\int q\cdot e^{\int p\,dt}\,dt\right)$

$\to y=e^{2\ln|t|}\cdot\left(c+\left(-\frac{1}{t}\right)\right)=t^2\left(c-\frac{1}{t}\right)=ct^2-t$, $c\in\mathbb{R}$

$\boxed{x(t)=y(t)^2=(ct^2-t)^2 \ ,\ c\in\mathbb{R}}$

6 Рикатијева ДЈ

$x'=p(t)x^2+q(t)x+r(t)$, $p,q,r:(a,b)\to\mathbb{R}$ непр.

$p\equiv 0\to$ ЛИН

$r\equiv 0\to$ БЕР $\alpha=2$

СМЕНА: $x_p(t)$ - једно партикуларно решење

$x(t)\rightsquigarrow y(t)$ $x(t)=x_p(t)+\frac{1}{y(t)}$ ⇝ ЛИН $x'=x_p'-\frac{y'}{y^2}$;

(2) а) $t(2t-1)\cdot x'+x^2-(4t+1)x+4t=0$

б) $x'=\frac{2\sin t-x^2\sin t\cos^2 t}{\cos^2 t}$

а) полиноми $x_p=a$, $x_p=at+b$, $x_p=at^2+bt+c$, ...

в) $\frac{2}{t^2}\leftrightsquigarrow x'\leftrightarrow x^2$

$x_p'=-\frac{a}{t^2}$ $\frac{2}{t^2}$

$x_p^2=\frac{a^2}{t^2}$

б) $x' = \frac{\ldots}{\cos^2 t}$

в) $x' + x^2 + \frac{4x}{t} + \frac{2}{t^2} = 0$

в) $\frac{2}{t^2} \leftrightarrow x' \leftrightarrow x^2$

$x_p = \frac{a}{t}$

$x_p' = -\frac{a}{t^2}$

$x_p^2 = \frac{a^2}{t^2}$

$4\frac{x_p}{t} = \frac{4a}{t^2}$

t^2

б) $x' = 2\cdot\frac{\sin t}{\cos^2 t} - x^2 \cdot \sin t$

$x_p = ?$

$\frac{\sin t}{\cos^2 t} \leftrightarrow x'$

$\left(\frac{a}{\cos}\right)' = -\frac{-\sin}{\cos^2}$

$\sin t \cdot \left(\frac{1}{\cos}\right)^2$

$x_p = \frac{a}{\cos t}$, $a \in \mathbb{R}$ ← Тражимо решење у овом облику

$x_p' = -a \cdot \frac{-\sin t}{\cos^2 t} = a \cdot \frac{\sin t}{\cos^2 t}$

$a \cdot \frac{\sin t}{\cos^2 t} = 2 \cdot \frac{\sin t}{\cos^2 t} - \frac{a^2}{\cos^2 t} \cdot \sin t \rightsquigarrow a = 2 - a^2$

$a^2 + a - 2 = 0$

$(a+2)(a-1) = 0$

$a = -2$
$a = 1$ } Било које $\rightsquigarrow x_p = \frac{1}{\cos t}$

$x = \frac{1}{\cos t} + \frac{1}{y} \Rightarrow x' = \frac{\sin t}{\cos^2 t} - \frac{y'}{y^2}$

$\frac{\sin t}{\cos^2 t} - \frac{y'}{y^2} = 2\frac{\sin t}{\cos^2 t} - \left(\frac{1}{\cos t} + \frac{1}{y}\right)^2 \sin t = 2\frac{\sin t}{\cos^2 t} - \left(\frac{1}{\cos^2 t} + \frac{2}{y\cos t} + \frac{1}{y^2}\right)\sin t$

$= 2\frac{\sin t}{\cos^2 t} - \frac{\sin t}{\cos^2 t} - \frac{\sin t}{y\cos t} - \frac{\sin t}{y^2} = \frac{\sin t}{\cos^2 t} - \frac{\sin t}{y\cos t} - \frac{\sin t}{y^2} \quad / \cdot y^2$

$-y' = -\frac{\sin t}{\cos t}\cdot y - \sin t$

$y' - \frac{\sin t}{\cos t} y = \sin t$ (лнн)

$\rightsquigarrow y(t)$, $x(t) = \frac{1}{\cos t} + \frac{1}{y(t)}$

Смена $\boxed{1}$ ③ $2t\,x''(t) + (1+\sqrt{t})\,x'(t) - x(t) = t + \sin(\sqrt{t})$, $t > 0$ $\quad x(t) \rightsquigarrow x(u)$

$u = \sqrt{t}$

ДЈ своди на $x'' + x' - 2x = 2u^2 + 2\sin u$ (касније ћемо знати да решимо)

$u = \sqrt{t}$

$\frac{du}{dt} = \frac{1}{2\sqrt{t}}$

$\frac{dx}{dt} \rightsquigarrow \frac{dx}{du}$

$x'(t) = \frac{dx}{dt} = \frac{dx}{du}\cdot\frac{du}{dt} = \frac{1}{2\sqrt{t}}\cdot\frac{dx}{du} = \frac{1}{2u}\cdot\frac{dx}{du}$

$$\frac{d^2x}{dt^2} \rightsquigarrow \frac{d^2x}{du^2}$$

$$x''(t) = \frac{d^2x}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}\left(\frac{1}{2u}\cdot\frac{dx}{du}\right) = \frac{d}{dt}\left(\frac{1}{2\sqrt t}\cdot\frac{dx}{du}\right) =$$

$$= \underbrace{\frac{d}{dt}\left(\frac{1}{2\sqrt t}\right)}\cdot\frac{dx}{du} + \frac{1}{2\sqrt t}\cdot\underbrace{\frac{d}{dt}}\left(\frac{dx}{du}\right) =$$

$$\frac{d}{dt} = \frac{du}{dt}\cdot\frac{d}{du}$$

$$\left(\frac{1}{2\sqrt t}\right)' = \left(\frac12 t^{-1/2}\right)' = -\frac14 t^{-3/2}$$

$$= \underbrace{-\frac14\cdot\frac{1}{t^{3/2}}}\cdot\frac{dx}{du} + \boxed{\frac{1}{2\sqrt t}\cdot\frac{du}{dt}}\cdot\boxed{\frac{d}{du}\left(\frac{dx}{du}\right)} =$$

$$= -\frac{1}{4t^{3/2}}\cdot\frac{dx}{du} + \boxed{\frac{1}{4t}}\cdot\boxed{\frac{d^2x}{du^2}} = -\frac{1}{4u^3}\cdot\frac{dx}{du} + \frac{1}{4u^2}\cdot\frac{d^2x}{du^2}$$

$$2t\,x''(t) + (1+\sqrt t)\,x'(t) - x(t) = t + \sin(\sqrt t)$$

$$2u^2\cdot\left(-\frac{1}{4u^3}\cdot\frac{dx}{du} + \frac{1}{4u^2}\cdot\frac{d^2x}{du^2}\right) + (1+u)\cdot\frac{1}{2u}\cdot\frac{dx}{du} - x = u^2 + \sin u$$

$$-\frac{1}{2u}\cdot\frac{dx}{du} + \frac12\cdot\frac{d^2x}{du^2} + \frac{1}{2u}\cdot\frac{dx}{du} + \frac12\cdot\frac{dx}{du} - x = u^2 + \sin u \ \Big/\cdot 2$$

$$\underbrace{\frac{d^2x}{du^2}}_{x''} + \underbrace{\frac{dx}{du}}_{x'} - 2x = 2u^2 + 2\sin u$$

2 $x(t) \longrightarrow y(t)$

$$y = f(x) \rightsquigarrow y' = f'(x)\cdot x'$$

(4) а) $t x^2 x' + x^3 = t\cos t \longrightarrow y(t) = (x(t))^3 \to y' = 3x^2x' \to t\cdot\frac{y'}{3} + y = t\cos t$ (ЛНЧ)

б) $x'\cos x = \frac{\sin x}{t} - \sin^2 x \longrightarrow y = \sin x \to y' = \cos x\cdot x' \to y' = \frac{y}{t} - y^2$ $\alpha=2$ (БЕР, РИК)

в) $x'\operatorname{tg} x + 4t^3\cos^3 x = 2t \longrightarrow x'\operatorname{tg} x = x'\cdot\frac{\sin x}{\cos x}$, $y = \cos x \to y' = -\sin x\cdot x' \to \frac{-y'}{y} + 4t^3\cdot y^3 = 2t \ /\cdot(-y)$

$y' - 4t^3 y^4 = -2yt$ $\alpha=4$ (БЕР)

г) $t e^x x' - 2t e^{x/2} = 4e^x \longrightarrow y = e^x$, $y' = e^x x' \to t\cdot y' - 2t\sqrt y = 4y$ $\alpha=\frac12$ (БЕР)

3 $x(t) \rightsquigarrow y(u)$

$x \longrightarrow y$

$t \rightsquigarrow u$

(5) Решити ДЈ: $\frac{dx}{dt} = \frac{x\cdot((\ln x)^2 + t)}{2t^{3/2}}$, $t>0$, $x>0$

уводећи смене $u = \sqrt t$, $y = \ln x$

$$\frac{dx}{dt} \rightsquigarrow \frac{dy}{du} = \boxed{\frac{dy}{dx}}\cdot\frac{dx}{dt}\cdot\boxed{\frac{dt}{du}} = \frac{2u}{e^y}\cdot\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{e^y}{2u}\cdot\frac{dy}{du}$$

$y = \ln x \to x = e^y$, $\frac{dy}{dx} = \frac{1}{x} = e^{-y}$

$u = \sqrt t \Leftrightarrow t = u^2$, $\frac{dt}{du} = 2u$

$$\frac{\cancel{e^y}}{\cancel{2u}}\cdot\frac{dy}{du} = \frac{\cancel{e^y}\,(y^2 + u^2)}{\cancel{2}u^{\cancel{3}\,2}}$$

$y = \ln x \to x = e^y$

$\frac{dy}{dx} = \frac{1}{x} = e^{-y}$

$u = \sqrt{t} \Rightarrow t = u^2$

$\frac{dt}{du} = 2u$

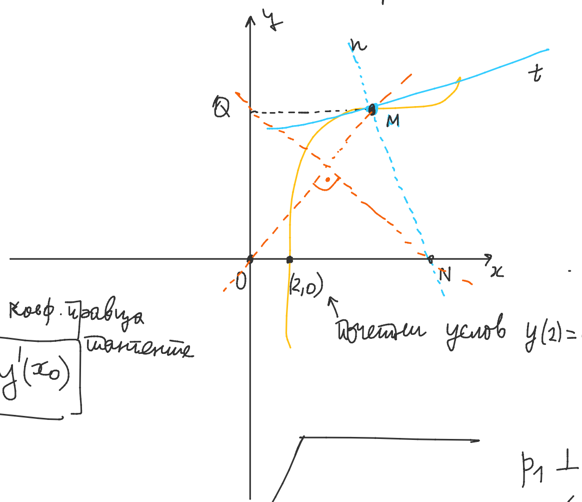
$\frac{\cancel{e^y}}{\cancel{2u}} \cdot \frac{dy}{du} = \frac{\cancel{e^y} \cdot (\dots)}{\cancel{2} u^{3\,2}}$

$\Rightarrow \frac{dy}{du} = \frac{y^2+u^2}{u^2} = \left(\frac{y}{u}\right)^2 + 1$ (ХОМ)

$y(u) \rightsquigarrow x(t) = \dots$

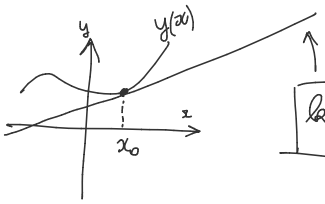
⑥ За (∀) тачку M криве c важи да се пројектује на y-осу у тачки Q, а нормала на c у M сече x-осу у тачки N. Ако је O коорд. почетак и важи $QN \perp OM$ и крива c пролази кроз $(2,0)$, одредити је.

$c: \underline{y(x)}$

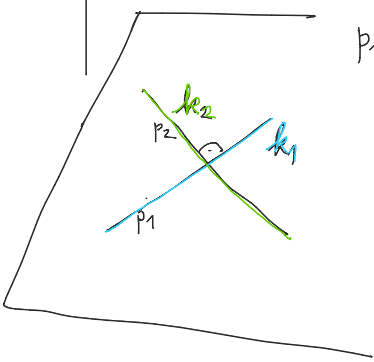


почетни услов $y(2) = 0$.

Где је овде ДЈ?



Коеф. правца тангенте $\boxed{k = y'(x_0)}$



$p_1 \perp p_2$

$\Leftrightarrow$

$k_1 \cdot k_2 = -1$

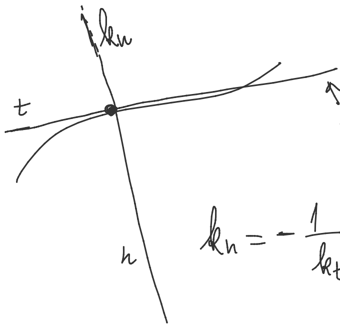
$k_2 = -\frac{1}{k_1}$

$M(x_0, y(x_0))$

$Q(0, y(x_0))$

$O(0,0)$

$N?$



$k_t = y'(x_0)$

$k_n = -\frac{1}{k_t} = -\frac{1}{y'(x_0)}$

n: пролази кроз $(x_0, y(x_0))$

k_n - коеф. пр.

$n: y = k_n x + n_n$

$y(x_0) = k_n \cdot x_0 + n_n \Rightarrow n_n = y(x_0) - k_n \cdot x_0 = y(x_0) + \frac{x_0}{y'(x_0)}$

$N(?, 0): \quad 0 = -\frac{1}{y'(x_0)} \cdot ? + \left(y(x_0) + \frac{x_0}{y'(x_0)}\right)$

$? = y'(x_0)\left(y(x_0) + \frac{x_0}{y'(x_0)}\right) = y(x_0) \cdot y'(x_0) + x_0$

$N\left(y(x_0) \cdot y'(x_0) + x_0, 0\right)$

$OM: \quad k_1 = \frac{y_M - y_O}{x_M - x_O} = \frac{y(x_0)}{x_0}$

$NQ: \quad k_2 = \frac{y_Q - y_N}{x_Q - x_N} = \frac{y(x_0) - 0}{0 - (y(x_0)y'(x_0) + x_0)} = -\frac{y(x_0)}{y(x_0)y'(x_0) + x_0}$

$OM \perp NQ$

$k_1 = -\frac{1}{k_2}$

$\frac{y(x_0)}{x_0} = \frac{y(x_0)y'(x_0) + x_0}{y(x_0)}$

$(\forall x_0) \Rightarrow$ ДЈ

$\frac{y}{x} = \frac{yy' + x}{y} \Rightarrow \boxed{xyy' + x^2 = y^2}$

$\forall x_0 \Rightarrow$ ДУ

x_0 - променлива

$y(x_0) \rightsquigarrow y$

$y'(x_0) \rightsquigarrow y'$

$$\frac{y}{x} = \frac{yy' + x}{y} \Rightarrow \boxed{x\underline{yy'} + x^2 = \underline{y}^2}$$

СМЕНА

$$y^2 = z$$

$$z' = 2yy'$$

$$\boxed{x \cdot \frac{z'}{2} + x^2 = z} \longrightarrow \text{лин}$$

$z(x) \rightsquigarrow y(x) \rightsquigarrow$. - . - -

$y(2) = 0$