2 - понедељак 19.10.2020.

Типови ДЈ

1 $x' = f(\underbrace{\alpha t + \beta x + \gamma})$, $f \in C(a,b)$, $\alpha, \beta \in \mathbb{R}\setminus\{0\}$, $\gamma \in \mathbb{R}$

смена: $x(t) \rightsquigarrow y(t)$

$$\left.\begin{aligned} & y(t) = \alpha t + \beta x(t) + \gamma \\ & y' = \alpha + \beta x' \Rightarrow x' = \frac{y' - \alpha}{\beta} \end{aligned}\right\} \rightsquigarrow \text{РП}$$

① а) $x' = x + 2t - 3$

б) $x' = (\underbrace{x+t})^2$ $\quad [f(\cdot) = \cdot^2 + 1]$

б) $y = x + t$

$y' = x' + 1$

$$y' - 1 = x' = (x+t)^2 = y^2 \Rightarrow y' = y^2 + 1 \Rightarrow \frac{dy}{y^2+1} = dt \,/\int$$

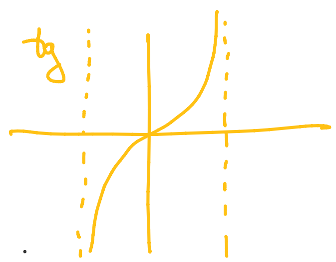
$$\arctan y = t + c, \quad c \in \mathbb{R} \quad /\,\mathrm{tg}$$

$$y = \mathrm{tg}(t+c), \quad c \in \mathbb{R}$$

$$x + t = \mathrm{tg}(t+c)$$

$$\boxed{x = \mathrm{tg}(t+c) - t, \quad c \in \mathbb{R}}$$

НАП:



$-\frac{\pi}{2} < t + c < \frac{\pi}{2}$

$-\frac{\pi}{2} - c < t < \frac{\pi}{2} - c$

∞ реш., а свако деф. на другачијем интервалу

+ решења не могу да се продуже, јер $\to +\infty$
(То продужењу $\rightsquigarrow$ предавања)

2 $x' = f\left(\underbrace{\frac{x}{t}}\right)$, $f \in C(a,b)$ (хомогена)

смена: $y(t) = \frac{x(t)}{t}$

$$\left. yt = x \,/' \Rightarrow \underbrace{x'} = (yt)' = y' \cdot t + y \cdot 1 = y + y't \right\} \rightsquigarrow \text{РП}$$

② а) $x' = e^{\frac{x}{t}} + \frac{x}{t}$

б) $x' = -\frac{x^2 + t^2}{2xt}$

б) $x' = -\frac{\dots}{2xt}$

в) $x' = \frac{x^2-2tx-t^2}{x^2+2tx-t^2}$

г) $t\sin\frac{x}{t}\cdot x' = x\cdot\sin\frac{x}{t}+t$

$f(\cdot) = \frac{\cdot^2-2\cdot-1}{\cdot^2+2\cdot-1}$

в) $$x' = \frac{x^2-2tx-t^2}{x^2+2tx-t^2} \overset{:t^2}{=} \frac{\left(\frac{x}{t}\right)^2-2\frac{x}{t}-1}{\left(\frac{x}{t}\right)^2+2\frac{x}{t}-1}$$

$$y=\frac{x}{t} \Rightarrow x = yt \,/' \Rightarrow x' = y't+y$$

$$y't+y = x' = \frac{\left(\frac{x}{t}\right)^2-2\frac{x}{t}-1}{\left(\frac{x}{t}\right)^2+2\frac{x}{t}-1} = \frac{y^2-2y-1}{y^2+2y-1}$$

$$\underset{\frac{dy}{dt}"}{y'}t = \frac{y^2-2y-1}{y^2+2y-1} - y = \frac{y^2-2y-1-y(y^2+2y-1)}{y^2+2y-1} = \frac{-y^3-y^2-y-1}{y^2+2y-1}$$

$=0?$

$$\frac{y^2+2y-1}{\boxed{y^3+y^2+y+1}}\,dy = -\frac{dt}{t} \;/\int$$

$$\int\frac{y^2+2y-1}{y^3+y^2+y+1}dy = -\int\frac{dt}{t} = -\ln|t|+c\,,\quad c\in\mathbb{R}$$

$$\frac{y^2+2y-1}{y^3+y^2+y+1} = \frac{y^2+2y-1}{(y+1)(y^2+1)} = \frac{A}{y+1}+\frac{By+C}{y^2+1} = \frac{A(y^2+1)+(y+1)(By+C)}{(y+1)(y^2+1)}$$

$y^2:\ 1 = A+B$

$y:\ 2 = B+C$

$1:\ -1 = A+C$

$(1)+(2)-(3):\ 2B = (1+2)-(-1) = 4$

$B=2$

$C=0$

$A=-1$

$$e^{\backslash}\ -\ln|t|+c = \int\dots = \int\frac{-1}{y+1}dy+\int\frac{2y}{y^2+1}dy = -\ln|y+1|+\ln|y^2+1| = \ln\left|\frac{y^2+1}{y+1}\right|$$

$$\frac{e^c}{|t|} = \left|\frac{y^2+1}{y+1}\right| \Rightarrow \frac{y^2+1}{y+1} = \frac{c_1}{t}\,,\quad c_1\in\mathbb{R}\setminus\{0\}$$

$$\frac{\left(\frac{x}{t}\right)^2+1}{\dots} = \frac{c_1}{\dots} \Rightarrow \boxed{\frac{x^2+t^2}{x+t} = c_1}$$

инициалното задание

$$\frac{\left(\frac{x}{t}\right)^2+1}{\frac{x}{t}+1} = \frac{c_1}{t} \Rightarrow \boxed{\frac{x^2+t^2}{x+t} = c_1}$$ (*)

задато

експлицитно зад:

$x(t) = ---$

$(y+1)\underbrace{(y^2+1)}_{\neq 0} = 0$? $\quad y+1=0$

(*) $x+t=0$

$y=-1 \Rightarrow \frac{x}{t} = -1 \Rightarrow \boxed{x=-t}$ решење ?

$$x' = -1 = \frac{t^2+2t^2-t^2}{t^2-2t^2-t^2} = \frac{2t^2}{-2t^2} = -1 \checkmark$$

НАП: на почетку делимо и са $x^2+2xt-t^2 \overset{?}{=} 0$

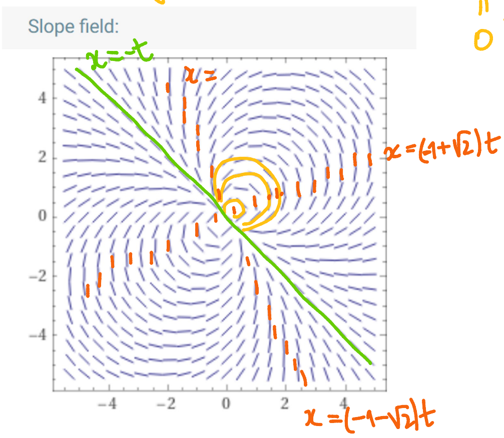
$\Rightarrow x' = \infty$? $\rightsquigarrow$ нагиб вертикалан

$x^2+2xt-t^2=0$, $\quad y=\frac{x}{t}$

$y^2+2y-1=0$

$y=-1\pm\sqrt{2}$

$x=(-1\pm\sqrt{2})t$



https://tinyurl.com/linkKaWA1

ако хоћемо ОР, не морамо овим да се бавимо

3 $\quad x' = f\left(\frac{a_1t+b_1x+c_1}{a_2t+b_2x+c_2}\right)$, $\quad f\in C(a,b)$

$\Downarrow$?

РП v хом

1° $c_1=c_2=0$: $\quad x' = f\left(\frac{a_1t+b_1x}{a_2t+b_2x}\right) = f\left(\frac{a_1+b_1\boxed{\frac{x}{t}}}{a_2+b_2\boxed{\frac{x}{t}}}\right) = g\left(\frac{x}{t}\right) \to$ хом.

2° $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \neq 0$

$x = v+\alpha \to v = x-\alpha$ $\qquad \alpha,\beta\in\mathbb{R}$

$\to t = u+\beta \to \frac{dt}{du}=1$ $\qquad x(t) \rightsquigarrow v(u) = x(u)-\alpha = x(t-\beta)-\alpha$

$$v' = \frac{dv}{du} = \underbrace{\boxed{\frac{dv}{dx}}}_{=1}\cdot\underbrace{\boxed{\frac{dx}{dt}}}_{=x'}\cdot\underbrace{\boxed{\frac{dt}{du}}}_{=1} = x'$$

$\alpha,\beta = ?$

$$\frac{a_1t+b_1x+c_1}{a_2t+b_2x+c_2}$$

$$a_1(u+\beta)+b_1(v+\alpha)+c_1 = a_1u+b_1v+(a_1\beta+b_1\alpha+c_1)$$
$$a_2(u+\beta)+b_2(v+\alpha)+c_2 = a_2u+b_2v+(a_2\beta+b_2\alpha+c_2)$$
$= 0$

α,β бирамо тд. су решења система $\quad a_1\beta+b_1\alpha+c_1 = 0$

$a_2\beta + b_2\alpha + c_2 = 0$

3° $\det = 0$ $\qquad \det \neq 0 \Rightarrow \exists_1 \alpha, \beta \rightsquigarrow$ ХОМ

$a_1 b_2 - a_2 b_1 = 0$

$a_1 b_2 = a_2 b_1$

3.1° неки од њих $= 0 \rightsquigarrow$ прав

3.2° $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = k \Rightarrow \dfrac{a_1 t + b_1 x + c_1}{a_2 t + b_2 x + c_2} = \dfrac{k a_2 t + k b_2 x + c_1}{a_2 t + b_2 x + c_2} = \dfrac{k\boxed{(a_2 t + b_2 x)} + c_1}{\boxed{(a_2 t + b_2 x)} + c_2}$

своди се на сл. $\boxed{1}$

(3) а) $(x+2t-2)x' = x - t - 1$

б) $x' = \dfrac{t+x+4}{t+x-6}$

а) $x' = \dfrac{x-t-1}{x+2t-2}$, $x + 2t - 2 = 0$?

$x = -2t + 2$

$0 \cdot \underbrace{x'}_{-2} = -2t + 2 - t - 1 = -3t + 1$ ✗

$\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = 2 - (-1) = 3 \neq 0$

$x = v + \alpha$

$t = u + \beta$

$\left.\begin{aligned} 1 - \alpha - \beta - 1 &= 0 \\ 1 - \alpha + 2\beta - 2 &= 0 \end{aligned}\right\} -$

$-\beta - 2\beta - 1 + 2 = 0$

$-3\beta = -1$

$\beta = \frac{1}{3} \Rightarrow \alpha = \frac{4}{3}$

$v' = \dfrac{dv}{du} = \underbrace{\dfrac{dv}{dx}}_{1} \cdot \boxed{\dfrac{dx}{dt}} \cdot \underbrace{\dfrac{dt}{du}}_{1} = x'$

$v' = \dfrac{\left(v + \frac{4}{3}\right) - \left(u + \frac{1}{3}\right) - 1}{\left(v + \frac{4}{3}\right) + 2\left(u + \frac{1}{3}\right) - 2} = \dfrac{v - u}{v + 2u}$ (ХОМ)

$\dfrac{v}{u} = w(u) \Rightarrow v = u \cdot w \Rightarrow v' = w + u \cdot w'$

$w + u \cdot w' = \dfrac{w-1}{w+2}$

$u \cdot w' = \dfrac{w-1}{w+2} - w = \dfrac{w - 1 - w(w+2)}{w+2} = \dfrac{-w^2 - w - 1}{w+2}$

$w^2 + w + 1 = 0$? ✗ $\qquad \dfrac{w+2}{w^2+w+1} dw = -\dfrac{du}{u} \quad / \int \ldots$

б) $x' = \dfrac{\boxed{t+x}+4}{\boxed{t+x}-6}$ (чак није $\boxed{3}$, него је пре свега $\boxed{1}$)

б) $x' = \dfrac{\boxed{t+x}+4}{\boxed{t+x}-6}$ (чак није [3], него је пре свега [1])

$t+x \neq 6$ због поставке

$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$

смена: $y = x+t \Rightarrow y' = x'+1$

$y' - 1 = \dfrac{y+4}{y-6}$ ...

4 Линеарна ДЈ 1. реда

$p, q : (a,b) \to \mathbb{R}$ непр.

$$x' + p(t)x = q(t)$$

$$\xrightarrow{\text{предавања}} x(t) = e^{-\int p\,dt} \cdot \Big(c + \underbrace{\int e^{\int p\,dt} \cdot q\,dt}\Big), \; c \in \mathbb{R}$$

$\int p\,dt = \int_{t_0}^{t} p(u)\,du$ ↳ исти на оба места

(4) а) $tx' - x = t^3$

б) $x' + x = \dfrac{1}{1+e^{2t}}$

в) $x' - 2xt = 6te^{t^2}$

г) $tx' + ax + t^n = 0$, $a \in \mathbb{R}, n \in \mathbb{N}$

$q(t) \equiv 0 \to$ хом. ЛДЈ1.р.

$q(t) \not\equiv 0 \to$ нехомогена ЛДЈ1.р.

б) $x' + \underset{p(t)=1}{x} = \underbrace{\boxed{\dfrac{1}{1+e^{2t}}}}_{=q(t)}$

$\int p\,dt = \int dt = t \, (+c)$

$\int e^{\int p\,dt} \cdot q\,dt = \int e^t \cdot \dfrac{1}{1+e^{2t}}\,dt = \int \dfrac{du}{1+u^2} = \operatorname{arctg} u \, (+c) = \operatorname{arctg}(e^t) \, (+c)$

$e^t = u$

$du = e^t dt$

ОР: $x(t) = e^{-t} \cdot \big(c + \operatorname{arctg}(e^t)\big)$, $c \in \mathbb{R}$

$= \underbrace{c \cdot e^{-t}}_{\text{реш. хом. дела}} + e^{-t} \cdot \operatorname{arctg}(e^t)$

а) $tx' - x = t^3 \;/ : t$

$x' \boxed{-\dfrac{1}{t}} x = \boxed{t^2}$

$p(t) = -\dfrac{1}{t}$ итд.

$q(t) = t^2$

(5) [illegible] $\sin^2 t$ [illegible] $x(t)=0$

$y(t) = t^-$

⑤ Нађи реш. Д.J. $x'\sin t - x\cos t = -\frac{\sin^2 t}{t^2}$ уз. $\lim\limits_{x\to\infty} x(t)=0$

I начин: линеарна $x' - \operatorname{ctg} t \cdot x = -\frac{\sin t}{t^2}$ ...

II начин: $x'\sin t - x\cos t = x'\cdot\sin t - x\cdot(\sin t)'$

$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

$$\frac{x'\sin t - x\cos t}{(\sin t)^2} = \left(\frac{x}{\sin t}\right)'$$

$$x'\sin t - x\cos t = -\frac{\sin^2 t}{t^2} \quad / : \sin^2 t$$

$$\left(\frac{x}{\sin t}\right)' = -\frac{1}{t^2} \quad / \int$$

$$\frac{x}{\sin t} = \frac{1}{t} + C \Rightarrow \boxed{x = C\sin t + \frac{\sin t}{t}, \quad C\in\mathbb{R}}$$

$\lim\limits_{x\to\infty} x(t) = 0$: $\frac{\sin t}{t} \xrightarrow[t\to\infty]{} 0$

$C\sin t \to 0$? $\Rightarrow \boxed{C=0}$

ПР: $\boxed{x(t) = \frac{\sin t}{t}}$