2 - петак 23.10.2020.

Типови ДЈ

1) $x' = f(\alpha t + \beta x + \gamma)$, $\alpha, \beta \in \mathbb{R}\setminus\{0\}$, $\gamma \in \mathbb{R}$, $f \in C(a,b)$

СМЕНА: $x(t) \rightsquigarrow y(t)$

$$\left.\begin{aligned} &y(t) = \alpha t + \beta x(t) + \gamma \\ &y' = \alpha + \beta x' \Rightarrow x' = \frac{y' - \alpha}{\beta} \end{aligned}\right\} \rightsquigarrow \text{РП}$$

(1) а) $x' = x + 2t - 3$

б) $x' = (x+t)^2$ $\quad [f(\bullet) = \bullet^2]$

б) $y = x + t$

$y' = x' + 1 \Rightarrow x' = y' - 1$

$x' = (t+x)^2 \Rightarrow y' - 1 = y^2$

$\frac{dy}{dt} = y' = y^2 + 1$

$\frac{dy}{y^2+1} = dt \quad / \int$

$\operatorname{tg} \Big/ \ \operatorname{arctg} y = t + c, \quad c \in \mathbb{R}$

$y = \operatorname{tg}(t+c)$

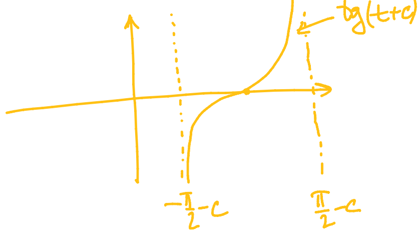
$x + t = \operatorname{tg}(t+c) \Rightarrow \boxed{x = \operatorname{tg}(t+c) - t, \ c \in \mathbb{R}}$

НАП: дефинисаност решења

$t + c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow t \in \left(-\frac{\pi}{2} - c, \frac{\pi}{2} - c\right)$

∞ много решења, свако решење је деф. на разл. интервалу

+решења не могу да се продуже



$\to +\infty$ верт. асимпт.

Т о продужењу решења (предавање)

2) $x' = f\left(\frac{x}{t}\right)$, $f \in C(a,b)$ (хомогена)

СМЕНА: $y(t) = \frac{x(t)}{t}$ $\quad (y' = \dots)$

$$\left. x = yt \ /' \Rightarrow x' = (yt)' = y' \cdot t + y \cdot (t)' = y't + y \right\} \rightsquigarrow \text{РП}$$

(2) а) $x' = e^{\frac{x}{t}} + \frac{x}{t}$

б) $x' = -\frac{x^2 + t^2}{2tx}$

в) $x' = \dfrac{x^2-2xt-t^2}{x^2+2xt-t^2}$

г) $t\sin\frac{x}{t}\cdot x' = x\sin\frac{x}{t} + t$

$$\left[f(z) = \frac{z^2-2z-1}{z^2+2z-1} \right]$$

в) $x' = \dfrac{x^2-2xt-t^2}{x^2+2xt-t^2} \Big| : t^2 = \dfrac{\left(\frac{x}{t}\right)^2 - 2\frac{x}{t} - 1}{\left(\frac{x}{t}\right)^2 + 2\frac{x}{t} - 1}$

$0 \neq$

$$\frac{x}{t} = y \Rightarrow x = yt \,/' \Rightarrow x' = y't + y$$

$$y't + y = \frac{y^2-2y-1}{y^2+2y-1}$$

$$\frac{dy}{dt}\cdot t = y't = \frac{y^2-2y-1-y(y^2+2y-1)}{y^2+2y-1} = \frac{-y^3-y^2-y-1}{y^2+2y-1}$$

$$\frac{y^2+2y-1}{y^3+y^2+y+1}dy = \frac{-dt}{t} \quad \Big/ \int$$

$$-\ln|t| + c = \int \frac{y^2+2y-1}{y^3+y^2+y+1}dy$$

$$\frac{y^2+2y-1}{y^3+y^2+y+1} = \frac{A}{y+1} + \frac{By+C}{y^2+1} = \frac{A(y^2+1)+(By+C)(y+1)}{(y+1)(y^2+1)} = \frac{y^2(A+B)+y(B+C)+(A+C)}{(y+1)(y^2+1)}$$

$$y^3+y^2+y+1 = (y^2+1)(y+1)$$

$$\left.\begin{aligned} 1 &= A+B \\ 2 &= B+C \\ -1 &= A+C \end{aligned}\right\} + \quad 2(A+B+C) = 2$$

$A+B+C = 1$

$C = 0,\ A = -1,\ B = 2$

$$\int \frac{y^2+2y-1}{y^3+y^2+y+1}dy = \int -\frac{dy}{y+1} + \int \frac{2y\,dy}{y^2+1} = -\ln|y+1| + \ln|y^2+1| = \ln\left|\frac{y^2+1}{y+1}\right|$$

$$-\ln|t| + c = \ln\left|\frac{y^2+1}{y+1}\right| \qquad e^c \in \mathbb{R}_+$$

$e^{\cdot} \backslash$

$$\frac{e^c}{|t|} = \left|\frac{y^2+1}{y+1}\right| \Rightarrow \frac{y^2+1}{y+1} = \frac{c_1}{t}, \quad c_1 \in \mathbb{R}\setminus\{0\},\ c_1 = \pm e^c$$

$(c_1 = 0 \Rightarrow y^2+1 = 0 \ \text{ϟ})$

$y = \frac{x}{t}$

$$\frac{\left(\frac{x}{t}\right)^2+1}{\frac{x}{t}+1} = \frac{c_1}{t} \Rightarrow \boxed{\frac{x^2+t^2}{x+t} = c_1,\ c_1 \in \mathbb{R}\setminus\{0\}} \ (1)$$

имплицитно решение

експлицитно:

$x(t) = \ldots$

$x + t = 0$

$y^3+y^2+y+1 = 0$?

$(y+1)(y^2+1) = 0 \Rightarrow y = -1$? $\boxed{x = -t}$ (2) решение?

ОР: (1)+(2)

$(y+1)\underbrace{(y^2+1)}_{y\neq 0}=0 \Rightarrow y=-1$? $\boxed{x=-t}$ (2) решение?

$-1 = x' = \frac{t^2+2t^2-t^2}{t^2-2t^2-t^2} = \frac{2t^2}{-2t^2} = -1$ ✓

ОР: (1)+(2)

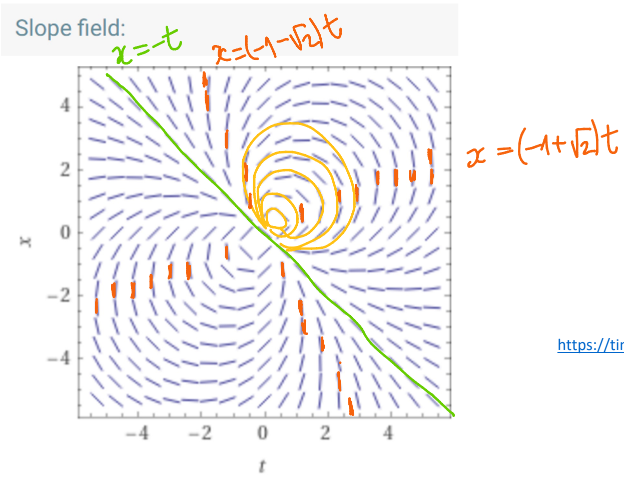
НАП: на точките делимо со $x^2+2xt-t^2=0$? } „$x' = \frac{---}{0}$" $\to \infty$" вертикални нагиб „$x'=\infty$"

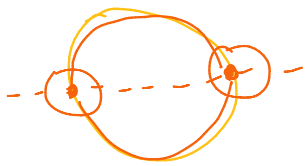
$y^2+2y-1=0$

$y=-1\pm\sqrt{2} \Rightarrow x=(-1\pm\sqrt{2})t$

$x=(-1-\sqrt{2})t$ ←

$x=(-1+\sqrt{2})t$ ←





https://tinyurl.com/linkKaWA1

$\boxed{3}$ $x' = f\left(\frac{a_1t+b_1x+c_1}{a_2t+b_2x+c_2}\right)$, $f\in C(a,b)$

РП v ХОМ

1° $c_1=c_2=0$: $x' = f\left(\frac{a_1t+b_1x}{a_2t+b_2x}\right) = f\left(\frac{a_1+b_1\frac{x}{t}}{a_2+b_2\frac{x}{t}}\right) = g\left(\frac{x}{t}\right) \to$ ХОМ

2° $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2-a_2b_1 \neq 0$

$v=x-\alpha$ ← $x=v+\alpha$ $\qquad \alpha,\beta\in\mathbb{R}$ $\qquad x(t) \longrightarrow v(u)=x(u)-\alpha = x(t-\beta)-\alpha$

$\frac{dv}{dx}=1$ $\qquad t=u+\beta \to \frac{dt}{du}=1$

$a_1t+b_1x+c_1 = a_1(u+\beta)+b_1(v+\alpha)+c_1 = a_1u+b_1v+\boxed{(a_1\beta+b_1\alpha+c_1)}=0$

$a_2t+b_2x+c_2 = a_2(u+\beta)+b_2(v+\alpha)+c_2 = a_2u+b_2v+\boxed{(a_2\beta+b_2\alpha+c_2)}=0$

Бираме α и β од: $\left.\begin{array}{l} a_1\beta+b_1\alpha+c_1=0 \\ a_2\beta+b_2\alpha+c_2=0 \end{array}\right\}$

ХОМ

$\det\neq 0 \Rightarrow \exists_1 \alpha,\beta$

$x(t) \rightsquigarrow v(u)$

$\frac{dx}{dt} \rightsquigarrow \frac{dv}{du}$

$\underbrace{\frac{dv}{du}}_{v'} = \underbrace{\frac{dv}{dx}}_{1}\cdot\underbrace{\frac{dx}{dt}}_{x'}\cdot\underbrace{\frac{dt}{du}}_{1} \Rightarrow v'=x'$

$\frac{dv}{du}=\frac{dx}{dt}$

3° $\det = 0 \Rightarrow a_1 b_2 = a_2 b_1$

3.1° неки од њих је нула $\Rightarrow$ трив.

3.2° $\frac{a_1}{a_2} = \frac{b_1}{b_2} = k \in \mathbb{R}$
$\left.\begin{array}{l} a_1 = k a_2 \\ b_1 = k b_2 \end{array}\right\}$
$$\frac{a_1 t + b_1 x + c_1}{a_2 t + b_2 x + c_2} = \frac{k a_2 t + k b_2 x + c_1}{a_2 t + b_2 x + c_2} = \frac{k(\boxed{a_2 t + b_2 x}) + c_1}{\boxed{a_2 t + b_2 x} + c_2}$$

своди на [1]

(3) а) $(x+2t-2)x' = x - t - 1$

б) $x' = \frac{t+x+4}{t+x-6}$

а) $x' = \frac{x-t-1}{x+2t-2}$

$\left[\begin{array}{l} x+2t-2=0? \\ \quad x = -2t+2 \\ \quad 0 \cdot x' = -2t+2-t-1 = -3t+1 \quad \times \end{array}\right]$

$$\det = \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = 2-(-1) = 3 \neq 0$$

$\left.\begin{array}{l} x = v + \alpha \\ t = u + \beta \end{array}\right\}$

$$\frac{dv}{du} = \frac{dx}{dt}$$

$$x - t - 1 = \underbrace{v - u}_{} + \underbrace{(\alpha - \beta - 1)}_{=0}$$
$$x + 2t - 2 = \underbrace{v + 2u}_{} + (\alpha + 2\beta - 2)$$

$\left.\begin{array}{l} \alpha - \beta = 1 \\ \alpha + 2\beta = 2 \end{array}\right\}$ $\beta = \frac{1}{3}$, $\alpha = \frac{4}{3}$

$$x' = \boxed{v' = \frac{v-u}{v+2u}} = \frac{\frac{v}{u} - 1}{\frac{v}{u} + 2} \quad \text{(хом)}$$

$$w(u) = \frac{v(u)}{u} \Rightarrow v = u \cdot w \,/' \Rightarrow v' = w'u + w = \frac{w-1}{w+2}$$

$$\Rightarrow \dots \quad \frac{w+2}{w^2+w+1}\, dw = -\frac{du}{u} \quad / \int$$

$=0$? $w^2 + w + 1 = (w + \frac{1}{2})^2 + \frac{3}{4} > 0$ $\times$

$w(u) \rightsquigarrow v(u) = u \cdot w(u) \rightsquigarrow x(t)$

б) $x' = \frac{t+x+4}{t+x-6}$ директно на [1], уместо [3]

$$\det = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

$=0$? $t+x-6 \neq 0$ због поставке

$t + x = y$

$y' = x' + 1$

$y' - 1 = \frac{y+4}{y-6}$ ...

[4] Линеарна ДЈ 1. реда

$p, q : (a,b) \to \mathbb{R}$ непр.

$$x' + p(t) \cdot x = g(t)$$

извод линеаран

линеаран од x

$g(t) \equiv 0 \to$ ХОМОГЕНА ЛДЈ 1.р.

$g(t) \not\equiv 0 \to$ нехомогена ЛДЈ 1.р.

t

линеаран од x

извод линеаран

предавања →

$$\boxed{x(t)=e^{-\int p\,dt}\left(c+\int e^{\int p\,dt}\cdot q\,dt\right),\ c\in\mathbb{R}}$$

$\int p\,dt=\int_{t_0}^{t}p(u)\,du$

→ прим. ф-ја од p
→ иста на оба места

④ а) $tx'-x=t^3$

б) $x'+x=\frac{1}{1+e^{2t}}$

в) $x'-2tx=6te^{t^2}$

г) $tx'+ax+t^n=0$, $a\in\mathbb{R}, n\in\mathbb{N}$

в) $p(t)=-2t$

$q(t)=6te^{t^2}$

$$\int p\,dt=\int -2t\,dt=-t^2\ (+c)$$

$$\int e^{-t^2}\cdot 6te^{t^2}\,dt=\int 6t\,dt=3t^2\ (+c)$$

$$\boxed{x(t)=e^{t^2}\cdot\left(c+3t^2\right),\ c\in\mathbb{R}}$$

$$=\underbrace{c\cdot e^{t^2}}_{\text{решење хомогене}}+\underbrace{3t^2e^{t^2}}_{\text{партикуларно реш.}}$$

а) $\boxed{t}\,x'-x=t^3$

$p(t)=-1$, $q(t)=t^3$ } X

$x'-\frac{1}{t}x=t^2$

$p(t)=-\frac{1}{t}$, $q(t)=t^2$ } ...

⑤ Нађи решење ДЈ. $x'\sin t-x\cos t=-\frac{\sin^2 t}{t^2}$ тд. $\lim_{t\to\infty}x(t)=0$.

I начин: $x'-\frac{\cos t}{\sin t}\cdot x=-\frac{\sin t}{t^2}$ линеарна → формула

II начин: $(x\sin t)'=x'\cdot\sin t+x\cdot\cos t$

$\left(\frac{x}{\sin t}\right)'=\frac{x'\sin t-x\cos t}{\ }$ ← ИДЕЈА

$$\left(\frac{x}{\sin t}\right)' = \frac{x'\sin t - x\cos t}{(\sin t)^2} \quad \leftarrow \text{ИДЕЈА}$$

$$x'\sin t - x\cos t = -\frac{\sin^2 t}{t^2} \quad / : \sin^2 t$$

$$\frac{x'\sin t - x\cos t}{(\sin t)^2} = -\frac{1}{t^2}$$

$$\left(\frac{x}{\sin t}\right)' = -\frac{1}{t^2} \quad / \int$$

$$\frac{x}{\sin t} = \frac{1}{t} + c \quad , c\in\mathbb{R} \Rightarrow \boxed{x(t) = \frac{\sin t}{t} + c\cdot\sin t,\ c\in\mathbb{R}}$$

$$\lim_{t\to\infty} x(t) = 0 : \quad \frac{\sin t}{t} \xrightarrow[t\to\infty]{} 0$$

$$\Rightarrow c\cdot\sin t \xrightarrow[t\to\infty]{} 0 \Rightarrow \boxed{c=0}$$

$$\Big\}\ \text{ПР}: \boxed{x(t) = \frac{\sin t}{t}}$$