Парцијалне ДЈ 1. реда

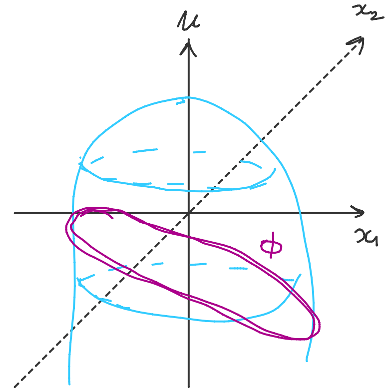
$$u(x_1,\dots,x_n)=? \leadsto F\Big(x_1,\dots,x_n,u,\underbrace{\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n}}_{1.\ \text{реда}}\Big)=0$$

квазилинеарна (КЛ): $\sum_{j=1}^{n} a_j(x_1,\dots,x_n,u)\frac{\partial u}{\partial x_j}=c(x_1,\dots,x_n,u)$

хомогена линеарна (ХЛ): $\sum_{j=1}^{n} a_j(x_1,\dots,x_n)\frac{\partial u}{\partial x_j}=0$ $(c=0)$

Кошијев проблем: нађи решење које садржи задату фигуру Φ.

$\Phi\subseteq\Gamma(u)$

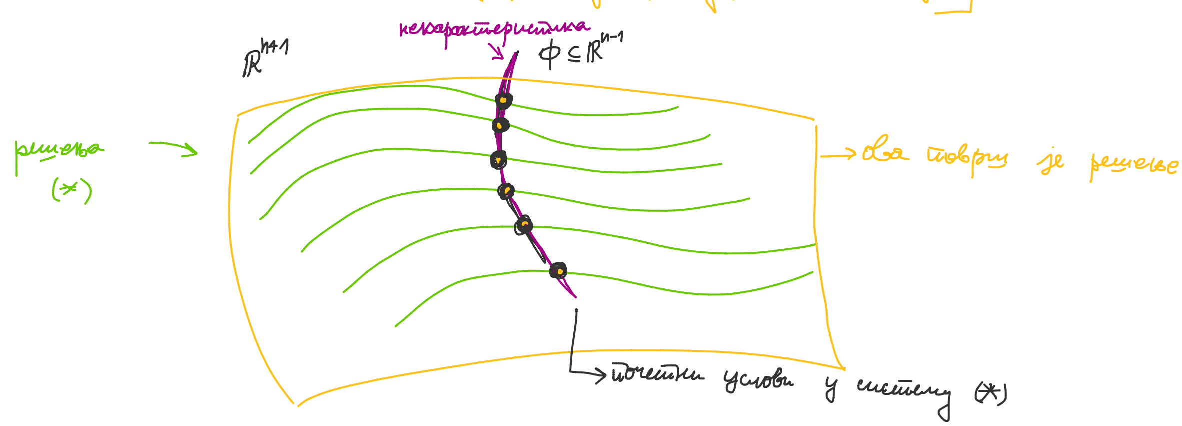


Метода карактеристика

(КЛ) $\Rightarrow$ систем карактеристика

$$x_j'(t)=a_j(x_1,\dots,x_n,u),\ \forall j$$
$$u'(t)=c(x_1,\dots,x_n,u) \qquad (*)$$

[(ХЛ) $x_j'(t)=a_j(x_1,\dots,x_n)$, $\forall j$]



① $\underbrace{u'_x+u'_y+u=1}_{(КЛ)}$ $\qquad$ $\underbrace{u(x,x+x^2)=\sin x}_{\text{Кошијев услов}}$, $x>0$ $\qquad$ [$u'_x=\frac{\partial u}{\partial x}$]

$c=1-u$

$u(x,y)=?$

$$\underbrace{1}_{a_1}\cdot\frac{\partial u}{\partial x}+\underbrace{1}_{a_2}\cdot\frac{\partial u}{\partial y}=\underbrace{1-u}_{c}$$

Прошле године:

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du}{1-u}$$

$$x'(t) = x' = 1 \Rightarrow x = t + c_1$$

$$y' = 1 \Rightarrow y = t + c_2$$

$$u' = 1-u \Rightarrow u = 1 + c_3 e^{-t}$$

(лин)

$u(x, x+x^2) = \sin x$

$C,\ \gamma(s) = (s, s+s^2, \sin s) \subseteq \Gamma(u)$

$x(t,s) = t + c_1(s)$, $\quad x_0(s) = s$

$y(t,s) = t + c_2(s)$, $\quad y_0(s) = s + s^2$, $\quad s > 0$

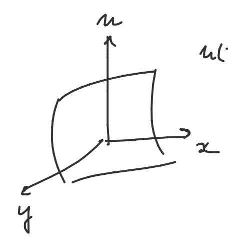
$u(t,s) = 1 + c_3(s) e^{-t}$, $\quad u_0(s) = \sin s$

$s = x_0(s) = x(0,s) = c_1(s)$

$s + s^2 = y_0(s) = y(0,s) = c_2(s)$

$\sin s = u_0(s) = u(0,s) = 1 + c_3(s) \Rightarrow c_3(s) = \sin s - 1$

$\big(x(t,s), y(t,s), u(t,s)\big) = \big(t+s,\ t+s+s^2,\ 1 + (\sin s - 1)e^{-t}\big) \to$ површ је решење



$u(x,y) = ?$

$y - x = (t+s+s^2) - (t+s) = s^2 \overset{s>0}{\Rightarrow} s = \sqrt{y-x}$

$t = x - s = x - \sqrt{y-x}$

$u = 1 + (\sin s - 1)e^{-t} = 1 + e^{-x+\sqrt{y-x}} \cdot \left(\sin\sqrt{y-x} - 1\right).$

Провера: $\quad u(x, x+x^2) = 1 + e^0 \cdot (\sin x - 1) = \sin x$

$u'_x + u'_y + u = 1$ ---

(2) $(y+u)\,u'_x + y\,u'_y = x - y$, $\quad u|_{y=1} = x+1$

$u(x,1) = x+1$

$(s, 1, s+1) \subseteq \Gamma(u)$

$x' = y + u$, $\quad x_0(s) = s$

$y' = y$, $\quad y_0(s) = 1$

$u' = x - y$, $\quad u_0(s) = s + 1$

$y' = y$,

$u' = x - y$, $\qquad u_0(s) = s+1$

$y = c_1 e^t \rightsquigarrow y(t,s) = c_1(s)\cdot e^t$

$t=0: \quad c_1(s) = y(0,s) = 1 \Big\} \quad y(t,s) = e^t$

$\begin{cases} x' = e^t + u \\ u' = x - e^t \end{cases}$ [домаћи: $w = x+u$]

$u = x' - e^t$

$u' = x'' - e^t$

$\Rightarrow x'' - e^t = x - e^t \Rightarrow x'' = x \quad (\lambda^2 - 1 = 0)$

$x = c_2 e^t + c_3 e^{-t}$

$u = x' - e^t = c_2 e^t - c_3 e^{-t} - e^t$

$t=0: \quad \left.\begin{array}{l} s = x(0,s) = c_2 + c_3 \\ s+1 = u(0,s) = c_2 - c_3 - 1 \end{array}\right\} \quad \begin{array}{l} c_2 = s+1 \\ c_3 = -1 \end{array}$

решење: $\big(\underset{x}{(s+1)e^t - e^{-t}},\ \underset{y}{e^t},\ \underset{u}{se^t + e^{-t}}\big)$

$t = \ln y \Rightarrow e^{-t} = e^{-\ln y} = \frac{1}{y}$

$x = (s+1)y - \frac{1}{y} \Rightarrow s = -1 + \frac{x + \frac{1}{y}}{y} = -1 + \frac{x}{y} + \frac{1}{y^2}$

$u(x,y) = \left(-1 + \frac{x}{y} + \frac{1}{y^2}\right)\cdot y + \frac{1}{y} = x - y + \frac{2}{y}$

Метода првих интеграла

идеја: у систему (*) наћи скупове где је решење и константно

↳ први интеграли система $\psi_1, \dots, \psi_{n-1}$

(3) $(x^2 - y^2 - z^2)\frac{\partial z}{\partial x} + 2xy\frac{\partial z}{\partial y} = 2xz$

$z(x,y)$ (КЛ)

↳ 2 интеграла ψ_1, ψ_2

прва

$x' = x^2 - y^2 - z^2$

$y' = 2xy$

$z' = 2xz$

$\frac{dy}{dz} = \frac{\frac{dy}{dt}}{\frac{dz}{dt}} = \frac{y'}{z'} = \frac{2xy}{2xz} = \frac{y}{z} \Rightarrow \frac{dy}{y} = \frac{dz}{z} \quad / \int$

$\rightarrow z' = 2xz \}$

$$\underline{\frac{dy}{dz}} = \frac{\frac{dy}{dt}}{\frac{dz}{dt}} = \frac{y'}{z'} = \frac{2xy}{2xz} = \underline{\frac{y}{z}} \quad \Rightarrow \quad \frac{dy}{y} = \frac{dz}{z} \quad / \int$$

(као да смо увели смену да тражимо $y(z) = ?$)

$$\ln|y| = \ln|z| + c_1$$

$\vdots$

$$\Rightarrow \frac{y}{z} = c_2 \rightsquigarrow \psi_1(x,y,z) = \frac{y}{z}$$

$$y = c_2 z \Rightarrow x' = x^2 - z^2(c_2^2+1)$$
$$z' = 2xz$$

$$\frac{dx}{dz} = \frac{\frac{dx}{dt}}{\frac{dz}{dt}} = \frac{x'}{z'} = \frac{x^2 - z^2(1+c_2^2)}{2xz} = \underbrace{\frac{x}{2z}}_{} - \underbrace{\frac{z(1+c_2^2)}{2x}}_{}$$

$x^{-1} = \frac{1}{x}$

Бернулијева $\alpha = -1$

смена: $u(z) = x(z)^2$

$\vdots$

$$x^2 = u(z) = cz - z^2(1 + c_2^2) \qquad c_2 \to \frac{y}{z}$$

$$x^2 = cz - z^2 - y^2 \Rightarrow c = \frac{x^2+y^2+z^2}{z} = \psi_2(x,y,z)$$

OP: $\varphi(\psi_1, \psi_2) = 0$, $\varphi \in C^1(\mathbb{R}^2)$

[OP: (кл) $\varphi(\psi_1, --) = 0$
(хл) $u = \varphi(\psi_1, --)$]

(4) $(4y-3z)\frac{\partial u}{\partial x} + (4x-2z)\frac{\partial u}{\partial y} + (2y-3x)\frac{\partial u}{\partial z} = 0$ (хл)

$u(x,y,z)$ (хл)

2 интеграла треба ψ_1, ψ_2
прва

$\Big\} (*)$

$x' = 4y - 3z$
$y' = 4x - 2z$
$z' = 2y - 3x$

[домаћи: методом карактеристика
$X' = AX \rightsquigarrow e^{tA}$ ---]

$$\alpha(4y-3z) + \beta(4x-2z) = 2y - 3x \qquad \alpha, \beta = ?$$

$$4\beta x + y(4\alpha) + z(-3\alpha - 2\beta) = 2y - 3x$$

$$\left.\begin{array}{l} 4\alpha = 2 \\ 4\beta = -3 \\ -3\alpha - 2\beta = 0 \end{array}\right\} \begin{array}{l} \alpha = \frac{1}{2} \\ \beta = -\frac{3}{4} \end{array}$$

$$\frac{1}{2}\underbrace{(4y-3z)}_{x'} - \frac{3}{4}\underbrace{(4x-2z)}_{y'} = \underbrace{2y-3x}_{z'} \quad / \cdot 4$$

$$2x' - 3y' = 4z' \quad / \int dt$$

[ПГ:
$$\frac{dx}{4y-3z} = \frac{dy}{4x-2z} = \frac{dz}{2y-3x}$$
$$\frac{\alpha dx + \beta dy}{\alpha(\ \ ) + \beta(\ \ )} = \frac{dz}{2y-3x}$$
$\vdots$]

x' y' z'

$$2x'-3y'=4z' \;/\int dt$$

$$2x-3y-4z=C \rightsquigarrow \psi_1=2x-3y-4z$$

$$\alpha\cdot x(4y-3z)+\beta\cdot y(4x-2z)=z\cdot(2y-3x) \quad ,\ \alpha,\beta=?$$

$$xy(4\alpha+4\beta)+xz(-3\alpha)+yz(-2\beta)=2yz-3xz$$

$$\left.\begin{aligned}4\alpha+4\beta&=0\\-3\alpha&=-3\\-2\beta&=2\end{aligned}\right\}\ \begin{aligned}\alpha&=1\\\beta&=-1\end{aligned}$$

$$xx'=\left(\frac{x^2}{2}\right)'$$

$$x\cdot x'-y\cdot y'=z\cdot z' \;/\int dt$$

$$\frac{x^2}{2}-\frac{y^2}{2}=\frac{z^2}{2}+c_1$$

$$x^2-y^2-z^2=2c_1=c_2 \implies \psi_2=x^2-y^2-z^2$$

ОР: $u=\varphi(\psi_1,\psi_2)$, $\varphi\in C^1(\mathbb{R}^2)$

⑤ $x(y^2-z^2)\frac{\partial u}{\partial x}-y(x^2+z^2)\frac{\partial u}{\partial y}+z(x^2+y^2)\frac{\partial u}{\partial z}=0$, $u|_{x=1}=(y+z)^2 \to$ површ S

→ Кошијев услов

Γ $u(1,y,z)=(y+z)^2$

$(1,y,z,(y+z)^2)\subseteq\Gamma(u)\subseteq\mathbb{R}^4$

ОР? $(X\cap)$, $u(x,y,z) \to 2$ прва инт.

$$x'=x(y^2-z^2)$$

$$y'=-y(x^2+z^2)$$

$$z'=z(x^2+y^2)$$

- $\alpha x(y^2-z^2)+\beta(-y(x^2+z^2))=z(x^2+y^2)$ ✗
- $\alpha\cdot x\cdot x(y^2-z^2)+\beta\cdot y\cdot(-y(x^2+z^2))=z\cdot z(x^2+y^2)$

$$\left.\begin{aligned}x^2y^2&:\ \alpha-\beta=0\\x^2z^2&:\ -\alpha=1\\y^2z^2&:\ -\beta=1\end{aligned}\right\}\ \alpha=\beta=-1$$

$$-x\cdot x'-y\cdot y'=z\cdot z' \;/\int dt$$

$$x^2+y^2+z^2=\psi_1$$

$$-x\cdot x' - y\cdot y' = z\cdot z' \,/ \int dt$$

$$x^2+y^2+z^2=\psi_1$$

- $\frac{\alpha}{x}\cdot x(y^2-z^2)+\frac{\beta}{y}\cdot(-y(x^2+z^2)) = \frac{1}{z}\cdot z(x^2+y^2)$

$$-\beta x^2+\alpha y^2+(-\alpha-\beta)z^2 = x^2+y^2$$

$\alpha=1$

$\beta=-1$

$$(\ln|x|)' = \frac{1}{x}\cdot x'$$

$$\frac{1}{x}\cdot x' - \frac{1}{y}y' = \frac{1}{z}z' \quad / \int dt$$

$$\ln|x| - \ln|y| = \ln|z| + c_1$$

$$\ln\left|\frac{x}{yz}\right| = c_1$$

$\vdots$

$$\frac{yz}{x} = c_2 \Rightarrow \psi_2 = \frac{yz}{x}$$

OP: $u=\varphi(\psi_1,\psi_2)$, $\varphi\in C^1(\mathbb{R}^2)$

Košijevo? $\rightsquigarrow \varphi = ?$

$$\overline{\psi_1} = \psi_1|_{x=1} = 1+y^2+z^2$$

$$\overline{\psi_2} = \psi_2|_{x=1} = yz$$

postavljamo na S: $u|_{x=1} = \varphi(\psi_1|_{x=1}, \psi_2|_{x=1})$

$$\underline{y}^2+2yz+\underline{z}^2 = (y+z)^2 = \varphi(1+y^2+z^2, yz)$$

$\to$ nađemo φ

$$\underbrace{(1+y^2+z^2)}_{\overline{\psi_1}}-1+2\cdot\underbrace{yz}_{\overline{\psi_2}} \Rightarrow \varphi(\psi_1,\psi_2) = \psi_1 - 1 + 2\psi_2$$

KP: $u = \psi_1+2\psi_2-1 = x^2+y^2+z^2+2\frac{yz}{x}-1.$