ЛДЈВРКК

$$F(t, x, x', \dots, x^{(n)}) = 0$$

- Јна линеарна ако је $F$ линеарно по $x, \dots, x^{(n)}$ ⇝ $a_0(t)\,x + a_1(t)x' + \dots + a_n(t)x^{(n)} = f(t)$
- $f \equiv 0$ ⇝ хомогена
- $a_0(t), \dots, a_n(t)$ константне ⇝ КК
- ОР: $x(t) = x_H(t) + x_P(t)$

  $x_H$ – опште решење хомогене
  $x_P$ – партикуларно нехомогене

- У случају ЛДЈВРКК: $a_n x^{(n)} + a_{n-1} x^{(n-1)} + \dots + a_1 x' + a_0 x = 0$ (хомогена)

$$a_n \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0$$ → карактеристична једначина

↳ има n решења у $\mathbb{C}$

→ реална нула $p$ вишеструкости $m$:

$e^{pt}, t\cdot e^{pt}, \dots, t^{m-1}e^{pt}$ ($m$ решења)

→ комплексна нула $\alpha \pm i\beta$ вишеструкости $m$: ($2m$ решења)

$e^{\alpha t}\cos\beta t, e^{\alpha t}\sin\beta t, te^{\alpha t}\cos\beta t, te^{\alpha t}\sin\beta t, \dots$

$t^{m-1}e^{\alpha t}\cos\beta t, t^{m-1}e^{\alpha t}\sin\beta t$

→ укупно n решења

Т: ОР хомогене је $x(t) = c_1x_1(t) + \dots + c_nx_n(t)$, $c_i \in \mathbb{R}$

(1) а) $x''' - 13x' - 12x = 0 \longrightarrow \lambda^3 - 13\lambda - 12 = 0 \longrightarrow (\lambda+1)\underbrace{(\dots)}_{\lambda^2 - \lambda - 12} = 0 \longrightarrow \lambda_1 = -1,\ \lambda_2 = -3,\ \lambda_3 = 4$

$\lambda = 1$ ×

$\lambda = -1$ ✓

$$\frac{1 \pm \sqrt{1 + 4 \cdot 12}}{2} = \frac{1 \pm 7}{2}$$

ОР: $x(t) = c_1e^{-t} + c_2e^{-3t} + c_3e^{4t}$, $c_1, c_2, c_3 \in \mathbb{R}$

б) $x''' - 7x'' + 16x' - 12x = 0$

б) $x''' - 7x'' + 16x' - 12x = 0$

$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$

$\lambda = 1$ ×

$\lambda = -1$ ×

$\lambda = 2$ ✓

$(\lambda - 2)\underbrace{(\lambda^2 - 5\lambda + 6)}_{(\lambda-3)(\lambda-2)} = 0$

$\rightarrow \lambda_1 = \lambda_2 = 2 \rightarrow e^{2t}, t \cdot e^{2t}$

$\lambda_3 = 3 \rightarrow e^{3t}$

ОР: $x(t) = c_1 e^{2t} + c_2 t e^{2t} + c_3 e^{3t}$, $c_i \in \mathbb{R}$

в) $x''' - 3x'' + 9x' + 13x = 0$

$\lambda^3 - 3\lambda^2 + 9\lambda + 13 = 0$

$\lambda = 1$ ×

$\lambda = -1$ ✓

$(\lambda + 1)(\lambda^2 - 4\lambda + 13)$

$D = 16 - 4 \cdot 13 = -36 < 0$

$\lambda_1 = -1 \longrightarrow e^{-t}$

$\lambda_{2/3} = \frac{4 \pm 6i}{2} = 2 \pm 3i \rightarrow e^{2t}\cos 3t, e^{2t}\sin 3t$

ОР: $x(t) = c_1 e^{-t} + c_2 e^{2t}\cos 3t + c_3 e^{2t}\sin 3t$

г) $x^{(6)} - 4x^{(5)} + 8x^{(4)} - 8x''' + 4x'' = 0$

$\lambda^6 - 4\lambda^5 + 8\lambda^4 - 8\lambda^3 + 4\lambda^2 = 0$

$\lambda^2(\lambda^4 - 4\lambda^3 + 8\lambda^2 - 8\lambda + 4) = 0$

$(\lambda^2 + a\lambda + b)(\lambda^2 + c\lambda + d) = (\lambda^2 - 2\lambda + 2)(\lambda^2 - 2\lambda + 2) = \underbrace{(\lambda^2 - 2\lambda + 2)^2}$

$D = 4 - 8 = -4$

$1 \pm i$

$\lambda = 1$ ×

$\lambda = -1$ ×

$\vdots$

$\lambda^3$: $a + c = -4$ ✓

$\lambda^2$: $b + d + ac = 8$

$\lambda$: $ad + bc = -8$

$1$: $bd = 4$

пробамо: $b = d = 2$ ; $a + c = -4$ ; $ac = 4$

$\Downarrow$

$a = c = -2$

$\lambda_1 = \lambda_2 = 0 \rightarrow e^{0t} = 1, t \cdot e^{0t} \rightarrow 1, t$

$\lambda_{3/4} = \lambda_{5/6} = 1 \pm i \rightarrow \ldots$

$(\alpha = \beta = 1)$

ОР: $x(t) = c_1 + c_2 t + c_3 e^t \cos t + c_4 e^t \sin t + c_5 t e^t \cos t + c_6 t e^t \sin t$, $c_i \in \mathbb{R}$

② Решити Кошијев проблем $x^{(III)} + x'' = 0$ (3. реда), $x(0) = 1$, $x^{(I)}(0) = 0$, $x^{(II)}(0) = 1$

$\lambda^3 + \lambda^2 = 0$

$\lambda^2(\lambda + 1) = 0$

$$OP: x(t) = c_1 + c_2 t + c_3 e^{-t}, \quad c_1, c_2, c_3 \in \mathbb{R}$$

$$x(0) = \underline{c_1 + c_3 = 1}$$

$$x'(t) = c_2 - c_3 e^{-t}$$

$$x'(0) = \underline{c_2 - c_3 = 0} \qquad \Rightarrow \qquad \begin{matrix} c_1 = 0 \\ c_2 = c_3 = 1 \end{matrix} \qquad \underline{x_K(t) = t + e^{-t}}$$

$$x''(t) = c_3 e^{-t}$$

$$x''(0) = \underline{c_3 = 1}$$

Нехомогена: $f \neq 0$

У неким случајевима знамо $\quad f(t) = e^{\alpha t} \cdot \left( P_n(t) \cdot \cos\beta t + Q_m(t) \cdot \sin\beta t \right)$

$P_n$ – пол. ст. $n$

$Q_m$ – пол. ст. $m$

$s$ – вишеструкост броја $\alpha \pm i\beta$ као решења карактеристичне једначине

$$x_p(t) = t^s \cdot e^{\alpha t} \cdot \left( R_k(t) \cdot \cos\beta t + T_k(t) \cdot \sin\beta t \right)$$

$R_k, T_k$ – пол. ст. $k$

$k = \max\{n, m\}$

3) a) $x''' - x'' + x' - x = t^2 + t \qquad (1, \pm i)$

$$OP: x(t) = \underbrace{x_H(t)}_{c_1 e^{t} + c_2 \cos t + c_3 \sin t,\ c_i \in \mathbb{R}} + x_p(t)$$

$$f(t) = t^2 + t$$

$\alpha = 0$

$\beta = 0 \quad (\sin\beta t = 0, \ \cos\beta t = 1)$

$$P_n(t) = t^2 + t: \quad \left. \begin{matrix} n = 2 \\ (m = 0) \end{matrix} \right\} k = \max\{m, n\} = 2$$

$$s = ? \qquad \alpha + i\beta = 0 + i \cdot 0 = 0 \Rightarrow s = 0$$

$$x_p(t) = R_2(t) = at^2 + bt + c \longrightarrow \begin{matrix} x_p' = 2at + b \\ x_p'' = 2a \\ x_p''' = 0 \end{matrix}$$

$$x_p''' - x_p'' + x_p' - x_p = t^2 + t$$

$$0 - 2a + (2at + b) - (at^2 + bt + c) = t^2 + t$$

$$t^2: \; -a = 1 \qquad t: \; 2a - b = 1 \qquad 1: \; -2a + b - c = 0 \quad \Rightarrow \quad a = -1, \; b = -3, \; c = -1$$

$$x_p(t) = -t^2 - 3t - 1$$

б) $x''' - x'' + x' - x = \underbrace{\cos t + 2e^t}$ → није у одг. облику

$$\underset{\text{лин. јед.}}{L}(x) = \underbrace{f_1(t)}_{x_{p_1}} + \underbrace{f_2(t)}_{x_{p_2}}$$

$$OP: \; x(t) = x_H(t) + \boxed{x_{p_1}(t) + x_{p_2}(t)}^{\,x_p(t)}$$

$(1, \pm i) \rightsquigarrow x_H$

$f_1(t) = \cos t$

$f_2(t) = 2e^t$

$f_1$: $\alpha = 0$, $\beta = 1$

$Q_m \equiv 0 \Rightarrow m = -\infty$, $n = 0$ $\Big\}$ $k = 0$

$R_0(t) = c_1$, $T_0(t) = c_2$ $\Big\}$ $\Rightarrow x_{p_1}(t) = t \cdot (c_1 \cos t + c_2 \sin t)$ $\ldots$ $c_1 = c_2 = -\frac{1}{4}$

$s = ?$ $\quad 0 \pm i 1 = \pm i \Rightarrow s = 1$

$f_2$: $\alpha = 1$, $\beta = 0$ $\Big\}$ $1 \pm i \cdot 0 = 1 \Rightarrow s = 1$

$n = 0$, $(m = 0)$ $\Big\}$ $k = 0$

$R_k(t) = c$

$(T_k(t) \ldots)$

$\Rightarrow x_{p_2}(t) = t \cdot e^t \cdot c$ $\;---\;$ $(c = ?)$

$x_{p_2}' = c e^t (1 + t)$

$x_{p_2}'' = c e^t (1 + t + 1)$

$x_{p_2}''' = c e^t (2 + t + 1)$

$$x_{p_2}''' - x_{p_2}'' + x_{p_2}' - x_{p_2} = 2e^t$$

$$c e^t \big((3 + t) - (2 + t) + (1 + t) - t\big) = 2e^t \Rightarrow c = 1 \Rightarrow x_{p_2}(t) = t e^t$$

$$OP: \; x(t) = \underbrace{c_1 e^t + c_2 \cos t + c_3 \sin t}_{x_H(t)} \underbrace{- \frac{t}{4}(\cos t + \sin t)}_{x_{p_1}(t)} + \underbrace{t e^t}_{x_{p_2}(t)}$$

в) $x'' - x = \sin^2 t$

$(\pm 1)$

$$\sin^2 t = \frac{1}{2}(1 - \cos 2t) = \underbrace{\frac{1}{2}}_{f_1} \underbrace{- \frac{1}{2} \cos 2t}_{f_2}$$

$f_1$: $\alpha = 0$

$f_2$: $\alpha = 0$

$f_1 \qquad f_2$

$f_1$: $\alpha=0$, $\beta=0$, $s=0$, $n=m=k=0$ $\Big\}$ $x_{p_1}(t)=C \Rightarrow C=-\frac{1}{2}$

$f_2$: $\alpha=0$, $\beta=2$, $s=0$, $n=m=k=0$ $\qquad x_{p_2}(t)=c_1\cos 2t+c_2\sin 2t$

г) $x''-4x'+5x=(\sin t+2\cos t)\cdot e^{2t}$

$(2\pm i)$

$\left.\begin{array}{l}\alpha=2\\ \beta=1\end{array}\right\} 2\pm i \Rightarrow s=1$, $n=m=k=0$ $\Big\}$ $x_p(t)=t\cdot e^{2t}(c_1\cos t+c_2\sin t)$ ...

д) $x''-2x'+x=\frac{e^t}{t}$ → није у одг. облику!

$(1 \times 2)$

$x_p(t)=e^t\cdot g(t) \qquad (g=?)$

$x_p'(t)=e^t(g(t)+g'(t))$

$x_p''(t)=e^t(g(t)+2g'(t)+g''(t))$

$\cancel{e^t}(g+2g'+g'')-2\cancel{e^t}(g+g')+\cancel{e^t}g=\frac{\cancel{e^t}}{t}$

$g''+\cancel{2g'}+\cancel{g}-\cancel{2g}-\cancel{2g'}+\cancel{g}=\frac{1}{t} \Rightarrow g''=\frac{1}{t} \Rightarrow g(t)=\int(\ln|t|+c_1)\,dt=c_2+\underline{c_1 t}+t\cdot\ln|t|\underline{-t}$

једно: $\left.\begin{array}{l}c_2=0\\ c_1=1\end{array}\right\} x_p(t)=e^t\cdot t\cdot\ln|t|$

ОР: $x(t)=c_1e^t+c_2te^t+e^t t\ln|t|$, $c_1,c_2\in\mathbb{R}$