Среда 30.12.2020.

① Наћи лимесе:

а) $\lim\limits_{x\to 0}\left(\dfrac{a^{x+1}+b^{x+1}+c^{x+1}}{a+b+c}\right)^{\frac{1}{x}}$, $a,b,c>0$

б) $\lim\limits_{x\to\infty}\dfrac{\log(x^2-x+1)}{\log(x^{10}+x^5+1)}$

в) $\lim\limits_{x\to 0}\left(\dfrac{\cos x}{\cos 2x}\right)^{\frac{1}{x^2}}$

г) $\lim\limits_{x\to 1}\dfrac{\sin(\pi x^{\alpha})}{\sin(\pi x^{\beta})}$, $\alpha,\beta\in\mathbb{R},\ \beta\neq 0$

д) $\lim\limits_{x\to\infty}\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)^{x}$

ђ) $\lim\limits_{x\to\infty}\left(x\left(\frac{\pi}{4}-\operatorname{arctg}\frac{x}{\sqrt{x^2+x}}\right)\right)$

а) $a^{x+1}\to a^1=a$

основа $\to \dfrac{a+b+c}{a+b+c}=1$

$1^{\pm\infty}$? $e^{\cdots}$

$$\lim_{x\to 0}\Bigg(1+\underbrace{\frac{a^{x+1}+b^{x+1}+c^{x+1}-a-b-c}{a+b+c}}_{\to 0,\ =t}\Bigg)^{\frac{1}{x}}=\lim_{x\to 0}\left\{(1+t)^{\frac{1}{t}}\right\}^{t\cdot\frac{1}{x}}=\lim_{x\to 0}e^{\frac{a^{x+1}-a+b^{x+1}-b+c^{x+1}-c}{a+b+c}\cdot\frac{1}{x}}$$

$\lim\limits_{t\to 0}(1+t)^{\frac{1}{t}}=e$

$$=e^{\lim\limits_{x\to 0}\frac{a\cdot\frac{(a^x-1)}{x}+b\frac{(b^x-1)}{x}+c\frac{(c^x-1)}{x}}{a+b+c}}=e^{\frac{a\ln a+b\ln b+c\ln c}{a+b+c}}=$$

$$=\left(a^a b^b c^c\right)^{\frac{1}{a+b+c}}.$$

$\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=?$

$a>0$

$$\boxed{\lim_{x\to 0}\frac{a^x-1}{x}=\ln a}$$

$$\lim_{x\to 0}\frac{\log(1+x)}{x}=1\Rightarrow\lim_{t\to 0}\frac{\log(\cancel{1}+e^t-\cancel{1})}{e^t-1}=1\Rightarrow\lim_{t\to 0}\frac{t}{e^t-1}=1\Rightarrow\lim_{t\to 0}\frac{e^t-1}{t}=1\Rightarrow\lim_{u\to 0}\frac{e^{\ln a\cdot u}-1}{\ln a\cdot u}=1$$

$x=e^t-1$

$x\to 0\Rightarrow t\to 0$

$t=\ln a\cdot u$

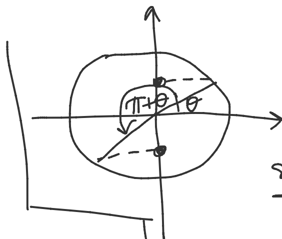
$t\to 0\Rightarrow u\to 0$

$e^{\ln a\cdot u}=\left(e^{\ln a}\right)^u=a^u$

$t\to 0 \Rightarrow u\to 0$

$$\Rightarrow \lim_{u\to 0}\frac{a^u-1}{u\cdot\ln a}=1 \Rightarrow \boxed{\lim_{u\to 0}\frac{a^u-1}{u}=\ln a}$$

г) $\displaystyle\lim_{x\to 1}\frac{\sin(\pi x^\alpha)}{\sin(\pi x^\beta)}$



$\sin(\pi+\theta)=-\sin\theta$

$\pi x^\alpha \xrightarrow[x\to 1]{} \pi$ $\qquad \dfrac{\sin t}{t}\xrightarrow[t\to 0]{} 1$

$\theta = x^\alpha\pi-\pi$

$\to \sin(x^\alpha\pi)=-\sin(x^\alpha\pi-\pi)$

$$\lim_{x\to 1}\frac{\sin(\pi x^\alpha)}{\sin(\pi x^\beta)}=\lim_{x\to 1}\frac{-\sin(\underbrace{\pi x^\alpha-\pi}_{\to 0})}{-\sin(\underbrace{\pi x^\beta-\pi}_{\to 0})}=\lim_{x\to 1}\frac{\overbrace{\dfrac{\sin(\pi x^\alpha-\pi)}{\pi x^\alpha-\pi}}^{\frac{\sin t}{t}\to 1}}{\underbrace{\dfrac{\sin(\pi x^\beta-\pi)}{\pi x^\beta-\pi}}_{\to 1}}\cdot\frac{\pi x^\alpha-\pi}{\pi x^\beta-\pi}$$

$$=\lim_{x\to 1}\frac{\pi x^\alpha-\pi}{\pi x^\beta-\pi}=\lim_{x\to 1}\frac{x^\alpha-1}{x^\beta-1}=\lim_{t\to 0}\frac{(1+t)^\alpha-1}{(1+t)^\beta-1}=\lim_{t\to 0}\frac{\overbrace{\dfrac{(1+t)^\alpha-1}{t}}^{\to\alpha}}{\underbrace{\dfrac{(1+t)^\beta-1}{t}}_{\to\beta}}=\frac{\alpha}{\beta}.$$

$x=t+1$

$x\to 1 \Rightarrow t\to 0$

Кратко о асимптотским релацијама:

- $\boxed{f(x)=o(g(x))}$, $x\to a$ $\iff$ $(\forall\varepsilon>0)$ $(\exists$ околина U од $a)$ тд. $|f(x)|\le\varepsilon\cdot|g(x)|$ на U.

ознака "мало о"

$g\neq 0 \Rightarrow \left|\dfrac{f(x)}{g(x)}\right|\le\varepsilon \iff \displaystyle\lim_{x\to a}\frac{f(x)}{g(x)}=0.$

$\cancel{o(g(x))=f(x)}$

пр. $x^3=o(x)$, $x\to 0$ $\iff$ $\displaystyle\lim_{x\to 0}\frac{x^3}{x}=\lim_{x\to 0}x^2=0.$

$x=10^{-100}$

$x^3 << x$

10^{-300} $\quad 10^{-100}$

$o(x^2) = o(x^3)$, $x \to +\infty$

$f \in o(x^2)_{, x\to\infty} \Rightarrow f \in o(x^3)$, $x \to \infty$

$\hookrightarrow \lim_{x\to\infty} \frac{f(x)}{x^2} = 0 \Rightarrow \lim_{x\to\infty} \frac{f(x)}{x^3} = 0 \Rightarrow f \in o(x^3), x\to\infty.$

пр. $\left.\begin{array}{l} f_1 = o(g(x)) \quad , x\to a \\ f_2 = o(g(x)) \quad , x\to a \end{array}\right\} \Rightarrow f_1 + f_2 = o(g(x))$, $x \to a$

$\left.\begin{array}{l} \lim_{x\to a} \frac{f_1(x)}{g(x)} = 0 \\ \lim_{x\to a} \frac{f_2(x)}{g(x)} = 0 \end{array}\right\} \lim_{x\to a} \frac{(f_1+f_2)(x)}{g(x)} = 0$

- $f(x) \sim g(x)$, $x \to a$

  „f се понаша као g"

  $\lim_{x\to a} \frac{f(x)}{g(x)} = 1$

$\sin x \sim x$, $x \to 0$

$e^x - 1 \sim x$, $x \to 0$

$\log(1+x) \sim x$, $x \to 0$

$1 - \cos x \sim \frac{1}{2}x^2$, $x \to 0$

$\cos x \sim 1 - \frac{1}{2}x^2$, $x \to 0$

Битно за рачун.

- $\sin x = x + o(x)$, $x \to 0$ $\Leftrightarrow$ $\sin x - x = o(x), x\to 0$ $\Leftrightarrow$ $\lim_{x\to 0} \frac{\sin x - x}{x} = 0 \Leftrightarrow \lim_{x\to 0} \frac{\sin x}{x} = 1.$

- $e^x - 1 = x + o(x)$, $x \to 0$

- $\log(1+x) = x + o(x)$, $x \to 0$

- $\cos x = 1 - \frac{1}{2}x^2 + o(x^2)$, $x \to 0$ $\longrightarrow$ $1 - \cos x = \frac{1}{2}x^2 + o(x^2)$ $\Leftrightarrow \lim_{x\to 0} \frac{1 - \cos x - \frac{1}{2}x^2}{x^2} = 0$

  не мења у −

- $(1+x)^\alpha = 1 + \alpha x + o(x), x \to 0$

пр. $\lim_{x\to 0} \frac{\sin 8x}{x} = \lim_{x\to 0} \frac{8x + o(8x)}{x} = \lim_{x\to 0} (8 + o(8)) = 8$

$\lim_{x\to 0} \frac{\log(1+x^2)}{1-\cos x}$

пр. $\lim\limits_{x\to 0}\frac{\sin 8x}{x}=\lim\limits_{x\to 0}\frac{8x+o(8x)}{x}=\lim\limits_{x\to 0}\left(8+o(8)\right)=8$

претпоставуваме дека се појавува x

$\boxed{o(8),\ x\to 0}$

$\lim\limits_{x\to 0}\frac{o(8)}{8}=0 \Leftrightarrow \lim\limits_{x\to 0} o(8)=0$

$\lim\limits_{x\to 0}\frac{\log(1+x^2)}{1-\cos x}$

$\lim\limits_{x\to 0} o(1)=0$

пр. $\lim\limits_{x\to 0}\frac{\log(1+x^2)}{1-\cos x}=\lim\limits_{x\to 0}\frac{x^2+o(x^2)}{\frac{1}{2}x^2+o(x^2)}\ \ {}^{:x^2}=\lim\limits_{x\to 0}\frac{1+o(1)}{\frac{1}{2}+o(1)}=\frac{1}{\frac{1}{2}}=2.$

$o(1)\to 0$

пр. $\lim\limits_{x\to\infty}\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)^x=\lim\limits_{x\to\infty} e^{x\cdot\log\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)}=\lim\limits_{x\to\infty} e^{x\cdot\log\left(1+\underbrace{\frac{1}{x}+o\left(\frac{1}{x}\right)}_{t}\right)}=\lim\limits_{x\to\infty} e^{x\cdot\left(\frac{1}{x}+o\left(\frac{1}{x}\right)\right)}=$

$=e^{\lim\limits_{x\to\infty}(1+o(1))}=e.$

$\sin\frac{1}{x}+\cos\frac{1}{x}=\frac{1}{x}+o\left(\frac{1}{x}\right)+1=\frac{1}{x}+1+o\left(\frac{1}{x}\right),\ x\to\infty$

$x\to\infty,\ \frac{1}{x}\to 0$

$\log(1+t)=t+o(t),\ t\to 0$

$\log\left(1+\frac{1}{x}+o\left(\frac{1}{x}\right)\right)=\frac{1}{x}+o\left(\frac{1}{x}\right)+o\left(\underbrace{\frac{1}{x}+o\left(\frac{1}{x}\right)}_{o\left(\frac{1}{x}\right)+o\left(\underbrace{o\left(\frac{1}{x}\right)}_{o\left(\frac{1}{x}\right)}\right)}\right)=\frac{1}{x}+o\left(\frac{1}{x}\right)$

$o\left(\frac{1}{x}\right)$

$o\left(\frac{1}{x}\right)$

$o(o(f))=o(f),\ x\to a$

$o(f)+o(f)=o(f)$

$o(c\cdot f)=o(f)$

② $f,g:\mathbb{R}\to\mathbb{R}$

f периодична со период $T>0$

g —||— $t>0$

$\forall x\in\mathbb{R}\quad f(x+T)=f(x)$

$g(x+t)=g(x)$

пр. $h(x)=\sin(x)$ $\quad \sin(x+2\pi)=\sin x$

$T=2\pi$

$\lim\limits_{x\to\infty}\left(f(x)-g(x)\right)=0.$

Докажете $f(x)=g(x),\ \forall x\in\mathbb{R}.$

пр. $h(x)=c\in\mathbb{R}$

$h(x+T)=c=h(x)$

$\lim_{x\to\infty} |f(x)-g(x)| = 0$.

Доказати $f(x)=g(x)$, $\forall x\in\mathbb{R}$.

пр. $h(x)=c\in\mathbb{R}$

$h(x+T)=c=h(x)$

1) доказујемо да f и g имају исти период

$$(\forall\varepsilon>0)(\exists M>0)(\forall x\geq M)\ |f(x)-g(x)|<\varepsilon$$

$\varepsilon>0$ произвољно. Узмемо M из ове деф

$$(\forall x\geq M)\ |f(x)-g(x)|<\varepsilon \quad (*)$$

$x\in\mathbb{R}$ узмемо произв. : $\exists k\in\mathbb{N}$ $\quad x+k\cdot t>M$ $\quad$ (Арх. аксиоме $\underbrace{k\cdot t}_{a} > \underbrace{M-x}_{b}$)

$$x+k\cdot t+T > x+k\cdot t>M$$

[g је t-периодична: $g(x)=g(x+k\cdot t)$, $k\in\mathbb{N}$]

$$|g(x+T)-g(x)| = |g(x+T+k\cdot t)-g(x+k\cdot t)| =$$

$$= |g(x+T+k\cdot t) - f(x+T+k\cdot t) + f(x+k\cdot t) - g(x+k\cdot t)| \overset{Н\Delta}{\leq}$$

$$\leq \underbrace{|g(\underbrace{x+T+kt}_{>M}) - f(\underbrace{x+T+kt}_{>M})|}_{<\varepsilon} + \underbrace{|f(\underbrace{x+kt}_{>M}) - g(\underbrace{x+kt}_{>M})|}_{<\varepsilon} < \varepsilon+\varepsilon = 2\varepsilon$$

из $(*)$

ε-произв. $\varepsilon\to 0$ $\quad g(x+T)=g(x)$. $\Rightarrow$ g је T-периодична

[исто докажемо да је f t-периодична]

$\Rightarrow$ g и f имају исти (неки) период (нпр. T или t) – означимо га са $\tau>0$

2) $x\in\mathbb{R}$ произв.

$$|f(x)-g(x)| = |f(x+l\cdot\tau)-g(x+l\cdot\tau)|<\varepsilon \quad / \lim_{\varepsilon\to 0} \quad \Rightarrow \quad f(x)=g(x).$$

$(*)$

$\varepsilon>0$ произв.

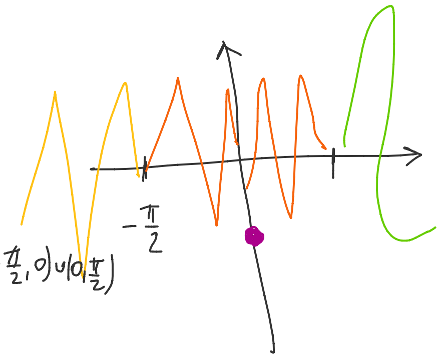
$M>0$ као у деф.

$(\exists l\in\mathbb{N})$ $\quad x+l\cdot\tau>M$

③ Нађи константе $A, B, C \in \mathbb{R}$ тд. је f непрекидна

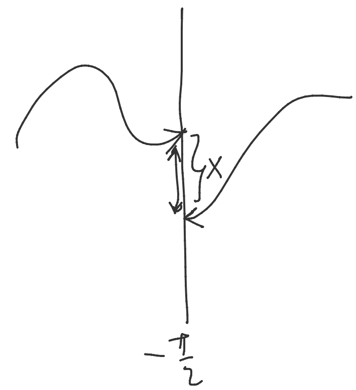
$$f(x) = \begin{cases} \dfrac{-2\sin x}{C}, & x \leq -\dfrac{\pi}{2} \\ , & x = 0 \\ A\dfrac{\sin x}{x} + Bx, & x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \setminus \{0\} \\ \cos x, & x \geq \dfrac{\pi}{2} \end{cases}$$

$\left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right)$



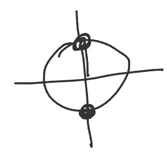
На $\left(-\infty, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \infty\right)$ је f непр.

у границама? $x = -\frac{\pi}{2}$, $x = 0$, $x = \frac{\pi}{2}$



у $x = -\frac{\pi}{2}$:

$$\lim_{x \to -\frac{\pi}{2}-} f(x) = \lim_{x \to -\frac{\pi}{2}+} f(x) = f\left(-\frac{\pi}{2}\right)$$



$$\lim_{x \to -\frac{\pi}{2}-} f(x) = f\left(-\frac{\pi}{2}\right) = -2\sin\left(-\frac{\pi}{2}\right) = 2.$$

$$\lim_{x \to -\frac{\pi}{2}+} f(x) = A \cdot \frac{\sin\left(-\frac{\pi}{2}\right)}{-\frac{\pi}{2}} + B\left(-\frac{\pi}{2}\right) = \frac{2}{\pi}A - \frac{\pi}{2}B$$

$$\boxed{\frac{2}{\pi}A - \frac{\pi}{2}B = 2}$$

у $x = \frac{\pi}{2}$:

$$\lim_{x \to \frac{\pi}{2}-} f(x) = \lim_{x \to \frac{\pi}{2}+} f(x)$$

$$\| $$

$$f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

$$\lim_{x \to \frac{\pi}{2}-} f(x) = A\frac{\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} + B\frac{\pi}{2} = \frac{2}{\pi}A + B\frac{\pi}{2}$$

$$\boxed{\frac{2}{\pi}A + \frac{\pi}{2}B = 0}$$

у $x = 0$: $\lim_{x \to 0-} f(x) = \lim_{x \to 0+} f(x) = f(0)$

$$f(0) = C$$

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0+} f(x) = \lim_{x \to 0} f(x) = \lim_{x \to 0}\left(A\underbrace{\frac{\sin x}{x}}_{1} + B\underbrace{x}_{0}\right) = A$$

$$\boxed{A = C}$$

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x) = \lim_{x\to 0} f(x) = \lim_{x\to 0} \Big(A \underbrace{\frac{\sin x}{x}}_{1} + B \underbrace{x}_{0}\Big) = A$$

$$\boxed{\begin{gathered} C = A = \frac{\pi}{2} \\ B = -\frac{2}{\pi} \end{gathered}}$$

(4) Дат је низ $(a_n)_{n\in\mathbb{N}}$ (Фекетова лема)

$(\forall m,n\in\mathbb{N})\quad 0 \le a_{m+n} \le a_m + a_n$

Доказати: $\left(\frac{a_n}{n}\right)_{n\in\mathbb{N}}$ конвергира

$$a_n = a_{(n-1)+1} \le \underbrace{a_{n-1}}_{(n-2)+1} + a_1 \le a_{n-2} + a_1 + a_1 \ --$$

$$\Rightarrow a_n \le n\cdot a_1 \Rightarrow \boxed{0 \le \frac{a_n}{n} \le a_1} \Rightarrow \frac{a_n}{n} \text{ ограничен}$$

$$\left\{\frac{a_n}{n} \,\middle|\, n\in\mathbb{N}\right\} \to \text{непразан и ограничен} \Rightarrow \exists\, C = \inf\left\{\frac{a_n}{n}\,\middle|\, n\in\mathbb{N}\right\}$$

Хоћемо: $\lim_{n\to\infty} \frac{a_n}{n} = C.$

$$\Leftrightarrow (\forall \varepsilon>0)(\exists n_0\in\mathbb{N})(\forall n\ge n_0)\ \underbrace{\left|\frac{a_n}{n} - C\right| < \varepsilon}$$

$$\underline{0 < \frac{a_n}{n} - C < \varepsilon} \quad \text{хоћемо!}$$

$\varepsilon > 0$ произв.

$$C = \inf \Rightarrow (\exists m\in\mathbb{N})\quad C \le \underset{\sim}{\frac{a_m}{m}} < C + \varepsilon$$

n-велики број (бар већи од m за сада)

$(\exists q, r)$ $\quad n = q\cdot m + r$

$q \in \mathbb{N}$

$r \in \{0, 1, \dots, m-1\}$

(делимо n са m са остатком)

$$\frac{a_n}{n} = \frac{a_{qm+r}}{n} \leq \frac{a_{qm} + a_r}{n} \leq \frac{q\cdot a_m + a_r}{n} \leq \frac{q\cdot a_m}{qm} + \frac{a_r}{n}$$

$n = qm + r \geq qm$

$$a_{qm} = a_{m + \dots + m} \leq a_m + a_m + \dots + a_m = q\cdot a_m$$

$$\Rightarrow \frac{a_n}{n} \leq \underbrace{\frac{a_m}{m}}_{< C+\varepsilon} + \frac{a_r}{n} < C + \varepsilon + \frac{a_r}{n}$$

$a_r \in \{a_1, a_2, \dots, a_{m-1}\}$, $A = \max\{a_1, \dots, a_{m-1}\}$

$(\exists n_1 \in \mathbb{N})$ $\quad n_1 \cdot \varepsilon > A$

$n_0 = \max\{m, n_1\}$

За свако $n \geq n_0$ шта важи?

$$n \geq n_1 \Rightarrow n\cdot\varepsilon > A \Rightarrow \frac{A}{n} < \varepsilon$$

$a_r \leq A$

$$\frac{a_n}{n} \leq C + \varepsilon + \frac{a_r}{n} \leq C + \varepsilon + \underbrace{\frac{A}{n}}_{<\varepsilon} < C + 2\varepsilon$$

$$C \leq \frac{a_n}{n} < C + 2\varepsilon, \quad \varepsilon \text{ произв.}$$

$$\varepsilon \to 0 \Rightarrow C \leq \frac{a_n}{n} \leq C \Rightarrow \lim_{n\to\infty} \frac{a_n}{n} = C.$$