Понедељак 29.03.2021.

Ако смо на интервалу $\Rightarrow$ прим. ф-је се разликују за конст

пр. скуп $\mathbb{R}\setminus\{0\} = (-\infty, 0) \cup (0, +\infty)$

$$F_1(x) = x^2 \qquad F_1'(x) = F_2'(x) = 2x, \text{ на } \mathbb{R}\setminus\{0\}$$

$$F_2(x) = \begin{cases} x^2, & x<0 \\ x^2+1, & x>0 \end{cases} \qquad F_1 - F_2 \neq const \text{ на } \mathbb{R}\setminus\{0\}$$

↳ није интервал

Лема: $\int (af(x) + bg(x))\,dx = a\cdot\int f(x)\,dx + b\cdot\int g(x)\,dx$, $a,b \in \mathbb{C}$

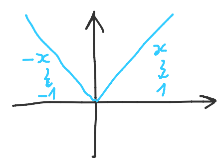
(линеарност)

$$\int f(x)\,dx = \int Re(f(x))\,dx + i\cdot\int Im(f(x))\,dx$$

Основни интеграли:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1 \qquad \left(\frac{x^{n+1}}{n+1}\right)' = \frac{1}{n+1}\cdot(x^{n+1})' = \frac{1}{n+1}\cdot(n+1)\cdot x^n = x^n$$

$$\int \frac{1}{x}\,dx = \log|x| + C \qquad (\log|x|)' = \frac{1}{|x|}\cdot(|x|)' \underset{x\neq 0}{=} \frac{1}{|x|}\cdot sgn(x) = \frac{1}{x}$$



$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{1}{\log a}\cdot a^x + C, \quad a>0, a\neq 1$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \left(\int \sin 2x\,dx \neq -\cos 2x + C\right)$$

$$\int \frac{1}{\cos^2 x}\,dx = tg\, x + C$$

$$\int \frac{1}{1+x^2}\,dx = arctg\, x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

① $\int \sin 2x\,dx = -\frac{1}{2}\cos 2x + C$

$$(-\cos 2x)' = -(-\sin 2x)\cdot 2 = 2\sin 2x$$

$$\Rightarrow \left(-\frac{1}{2}\cos 2x\right)' = \sin 2x$$

② $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^3 dx = \int \left(x^{3/2} + 3\cdot x\cdot x^{-1/2} + 3\cdot x^{1/2}\cdot x^{-1} + x^{-3/2}\right)dx = \int x^{3/2}\,dx + 3\cdot\int x^{1/2}\,dx + 3\cdot\int x^{-1/2}\,dx + \int x^{-3/2}\,dx =$

(2) $\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^3 dx = \int\left(x^{3/2}+3\cdot x\cdot x^{-1/2}+3\cdot x^{1/2}\cdot x^{-1}+x^{-3/2}\right)dx = \int x^{3/2}dx + 3\int x^{1/2}dx + 3\cdot\int x^{-1/2}dx + \int x^{-3/2}dx =$

$$= \left(\frac{x^{3/2+1}}{3/2+1}+C_1\right)+3\cdot\left(\frac{x^{1/2+1}}{1/2+1}+C_2\right)+3\cdot\left(\frac{x^{-1/2+1}}{-1/2+1}+C_3\right)+\left(\frac{x^{-3/2+1}}{-3/2+1}+C_4\right)$$

$$(a+b)^3=\sum_{k=0}^{3}\binom{3}{k}a^k b^{3-k}=a^3+3a^2b+3ab^2+b^3$$

$$= \frac{x^{5/2}}{5/2}+3\cdot\frac{x^{3/2}}{3/2}+3\cdot\frac{x^{1/2}}{1/2}+\frac{x^{-1/2}}{-1/2}+\underbrace{(C_1+3C_2+3C_3+C_4)}_{=C} =$$

$$= \frac{2}{5}x^{5/2}+\frac{1}{2}x^{3/2}+6x^{1/2}-2x^{-1/2}+C$$

(3) $\int \sin^2 x\,dx = \int\frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2}\int 1dx - \frac{1}{2}\int\cos 2x\,dx = \frac{x}{2}-\frac{1}{2}\cdot\frac{1}{2}\sin 2x + C = \frac{x}{2}-\frac{1}{4}\sin 2x + C.$

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$\left(\frac{1}{2}\sin 2x\right)' = \frac{1}{2}\cos 2x\cdot 2 = \cos 2x$

Запис. $\int 1dx = \int dx$

$\int\frac{1}{x}dx = \int\frac{dx}{x}$

Парцијална интеграција: $\int u(x)\cdot v'(x)\,dx = u(x)\cdot v(x) - \int u'(x)\,v(x)\,dx$ $\qquad du = u'\cdot dx$

(u, v – диференцијабилне) $\qquad \left(\int u\,dv = u\cdot v - \int v\,du\right)$ $\qquad (u\cdot v)' = u'v+uv' \,/\int \;\dots$

(4) $\int x\log x\,dx = \left|\begin{array}{l} u=\log x \\ dv = x\,dx\end{array} \Rightarrow \begin{array}{l} du=(\log x)'dx = \frac{dx}{x} \\ v=\frac{x^2}{2}\end{array}\right| = \frac{x^2}{2}\cdot\log x - \int\frac{x^2}{2}\cdot\frac{dx}{x} = \frac{x^2}{2}\log x - \int\frac{x}{2}dx = \frac{x^2}{2}\log x - \frac{x^2}{4}+C$

$x \to$ инт.

$\log x \to$ извод.

(5) $\int x^3e^x dx = \left|\begin{array}{l} u=x^3 \\ dv=e^x dx\end{array} \Rightarrow \begin{array}{l} du=3x^2dx \\ v=e^x\end{array}\right| = x^3\cdot e^x - \int 3x^2\cdot e^x dx = x^3\cdot e^x - 3\cdot\int x^2e^x dx = \left|\begin{array}{l} u=x^2 \\ dv=e^x dx\end{array} \Rightarrow \begin{array}{l} du=2x\,dx \\ v=e^x\end{array}\right| =$

$= x^3e^x - 3\cdot\left(x^2\cdot e^x - \int 2x\cdot e^x dx\right) = x^3\cdot e^x - 3x^2e^x + 6\int xe^x dx = \left|\begin{array}{l} u=x \\ dv=e^x dx\end{array} \Rightarrow \begin{array}{l} du=dx \\ v=e^x\end{array}\right| =$

$= x^3\cdot e^x - 3x^2e^x + 6\cdot\left(x\cdot e^x - \underbrace{\int e^x dx}_{=\,e^x+C}\right) = e^x\left(x^3-3x^2+6x-6\right)+C$

слично и за све $\int \underbrace{P(x)}\cdot e^x dx$ и $\int \underbrace{P(x)}\cdot\log x\,dx$

$\to$ полином

(6) $\int e^x\cdot\sin x\,dx = \left|\begin{array}{l} u=e^x \end{array} \Rightarrow \begin{array}{l} du=e^x dx \end{array}\right| = -\cos x\cdot e^x - \int e^x\cdot(-\cos x)\,dx = -\cos x\cdot e^x + \int e^x\cos x\,dx =$

⑥ $\int e^x \cdot \sin x \, dx = \left/ \begin{matrix} u = e^x \\ dv = \sin x \, dx \end{matrix} \Rightarrow \begin{matrix} du = e^x dx \\ v = -\cos x \end{matrix} \right/ = -\cos x \cdot e^x - \int e^x \cdot (-\cos x) dx = -\cos x \cdot e^x + \int e^x \cos x \, dx =$

$= \left/ \begin{matrix} u = e^x \\ dv = \cos x \, dx \end{matrix} \Rightarrow \begin{matrix} du = e^x dx \\ v = \sin x \end{matrix} \right/ = -\cos x \cdot e^x + \left(e^x \cdot \sin x - \int e^x \sin x \, dx \right) = e^x (\sin x - \cos x) - \int e^x \sin x \, dx$

$I = \int e^x \cdot \sin x \, dx \Rightarrow I = e^x (\sin x - \cos x) - I \Rightarrow I = \frac{1}{2} e^x (\sin x - \cos x) + C = \int e^x \sin x \, dx.$

слично и за $\int e^x \cos x \, dx$.

⑦ (6 на други начин)

$$\int e^{ax} dx = \frac{e^{ax}}{a} + C, \ a \neq 0$$

$$e^{ix} = \cos x + i \sin x$$

$R = \int e^x \cos x \, dx$

$I = \int e^x \sin x \, dx$

$C = C_1 + i C_2$

$$R + iI = \int e^x (\cos x + i \sin x) dx = \int e^x \cdot e^{ix} dx = \int e^{(1+i)x} dx \overset{a = 1+i}{=} \frac{e^{(1+i)x}}{1+i} + C = \frac{1-i}{2} e^x (\cos x + i \sin x) + C =$$

$$\left(\frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1-i}{2} \right)$$

$$= \frac{e^x}{2} (\cos x + \sin x) + i \cdot \frac{e^x}{2} (-\cos x + \sin x) + C$$

$\Rightarrow R = \frac{e^x}{2} (\cos x + \sin x) + C_1$

$I = \frac{e^x}{2} (\sin x - \cos x) + C_2$

<u>Смена променљиве</u>

$\int f(x) dx = F(x) + C$ на (a, b)

$\varphi : (\alpha, \beta) \to (a, b)$ непр. диф. (C^1) и НА

$\Rightarrow \int f \circ \varphi(t) \cdot \varphi'(t) \, dt = F \circ \varphi(t) + C$

⑧ $\int \cos 2x \, dx = \int \cos(t) \frac{dt}{2} = \frac{1}{2} \cdot \int \cos(t) dt = \frac{1}{2} \sin(t) + C = \frac{1}{2} \sin(2x) + C.$

$t = 2x \ / d$

$dt = 2 dx \Rightarrow dx = \frac{dt}{2}$

$\int \cos 2x \, dx = \int \cos 2x \cdot \frac{1}{2} d(2x) = \frac{1}{2} \sin(2x) + C$

Правило T:

$\int \cos 2t \, dt = ?$

$f(x) = \cos x$

$\int \cos x = \sin x + C \quad , \; F(x) = \sin x$

$\varphi(t) = 2t$

$\varphi'(t) = 2$

$$\Rightarrow \int f\circ\varphi(t)\cdot\varphi'(t)\,dt = F\circ\varphi(t) + C$$

$$\int \cos(2t)\cdot 2\,dt = \sin(2t) + C$$

$$2\int \cos 2t\,dt = \sin 2t + C$$

$$\Rightarrow \int \cos 2t\,dt = \frac{1}{2}\sin 2t + \tilde{C}$$

(9) $$\int \frac{x\,dx}{x^2+1} = \int \frac{\frac{dt}{2}}{t+1} = \frac{1}{2}\int\frac{dt}{t+1} = \frac{1}{2}\log|t+1| + C = \frac{1}{2}\log|x^2+1| + C = \frac{1}{2}\log(x^2+1) + C.$$

$x^2 = t \,/\, d \qquad d(x^2) = (x^2)'\,dx = 2x\,dx$

$2x\,dx = dt$

$\Rightarrow x\,dx = \frac{dt}{2}$

формално $u = t+1$, $du = dt$ $\quad \int\frac{dt}{t+1} = \int\frac{du}{u} = \log|u| + C = \log|t+1| + C$

$\varphi(t) = t^2$

може и: $\varphi(t) = t^2+1$

(10) $$\int \arcsin x\,dx = \Big/\; \begin{matrix} u = \arcsin x \\ dv = dx \end{matrix} \Rightarrow \begin{matrix} du = \frac{1}{\sqrt{1-x^2}}dx \\ v = x \end{matrix} \Big/ = x\cdot\arcsin x - \int\frac{x\,dx}{\sqrt{1-x^2}} = \Big/\; \begin{matrix} t = x^2 \\ dt = 2x\,dx \end{matrix} \Big/ = x\cdot\arcsin x - \int\frac{\frac{dt}{2}}{\sqrt{1-t}} =$$

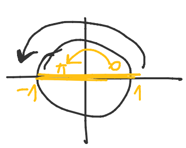
$$= x\cdot\arcsin x - \frac{1}{2}\int(\underbrace{1-t}_{})^{-1/2}dt = \Big/\; \begin{matrix} 1-t = u \\ du = -dt \end{matrix} \Big/ = x\cdot\arcsin x + \frac{1}{2}\int u^{-1/2}(+du) =$$

$$= x\arcsin x + \frac{1}{\cancel{2}}\cdot\frac{u^{1/2}}{\cancel{1/2}} + C = x\cdot\arcsin x + \sqrt{1-x^2} + C.$$

$$u^{1/2} = (1-t)^{1/2} = (1-x^2)^{1/2} = \sqrt{1-x^2}$$

(11) $$\int\sqrt{1-x^2}\,dx = \Big/\; \begin{matrix} x = \cos t \;, \; t\in[0,\pi] \Rightarrow \sin t \geq 0 \\ dx = -\sin t\,dt \end{matrix} \Big/ = \int\sqrt{1-\cos^2 t}\cdot(-\sin t\,dt) = -\int\underbrace{\sqrt{\sin^2 t}}_{|\sin t| = \sin t}\cdot\sin t\,dt = -\int\sin^2 t\,dt$$

$(a,b)\subseteq[-1,1]$



$$= -\frac{t}{2} + \frac{1}{4}\sin 2t + C = -\frac{\arccos x}{2} + \underbrace{\frac{1}{4}\sin(2\arccos x)}_{} + C$$

(чак $= \frac{1}{2}\sqrt{1-x^2}\cdot x$)

$x = \cos t$

$t = \arccos x$, где $t\in[0,\pi]$

(12) $$\int x^{2x}(1+\log x)\,dx = \frac{x^{2x}}{2} + C$$

$$x^{2x} = e^{2x\cdot\log x}$$

$$(x^{2x})' = \left(e^{2x\log x}\right)' = e^{2x\log x}\cdot(2x\log x)' = \boxed{2x^{2x}(\log x + 1)}$$

(13) $\int \ldots \quad / \; x = a\cdot \operatorname{sh} t \quad / \ldots \quad \int a^2\sqrt{1+\operatorname{sh}^2 t}\cdot\operatorname{ch} t\,dt =$

(13) $\int \sqrt{a^2+x^2}\,dx = \left/ \begin{matrix} x = a\cdot \operatorname{sh} t \\ dx = a\cdot \operatorname{ch} t\, dt \end{matrix} \right/ = \int \sqrt{a^2+a^2\operatorname{sh}^2 t}\cdot a\cdot \operatorname{ch} t\, dt = \int a^2\cdot\sqrt{1+\operatorname{sh}^2 t}\cdot \operatorname{ch} t\, dt =$

$a>0$

$\int e^{2t}dt = \frac{e^{2t}}{2}$

$= a^2\int\sqrt{\operatorname{ch}^2 t}\cdot \operatorname{ch} t\, dt = a^2\int \operatorname{ch}^2 t\, dt = a^2\int \frac{e^{2t}+e^{-2t}+2}{4}\,dt =$

$= \frac{a^2}{4}\Big(2t + \underbrace{\frac{e^{2t}}{2} - \frac{e^{-2t}}{2}}_{\operatorname{sh}(2t)}\Big) + C = \frac{a^2}{4}\left(2\operatorname{arsh}\frac{x}{a} + \operatorname{sh}\left(2\cdot \operatorname{arsh}\frac{x}{a}\right)\right) + C.$

$\operatorname{ch} x = \frac{e^x+e^{-x}}{2} \geq 0$

$\operatorname{sh} x = \frac{e^x-e^{-x}}{2}$

$\operatorname{ch}^2 x - \operatorname{sh}^2 x = 1$

$\Rightarrow 1+\operatorname{sh}^2 x = \operatorname{ch}^2 x$

$(\operatorname{ch} x)' = \frac{e^x-e^{-x}}{2} = \operatorname{sh} x$

$(\operatorname{sh} x)' = \frac{e^x+e^{-x}}{2} = \operatorname{ch} x$

$x = a\operatorname{sh} t$

$\frac{x}{a} = \operatorname{sh} t$

$t = \operatorname{arsh}\frac{x}{a}$

ареа-синус хиперболички

$\operatorname{arsh} x = \operatorname{arsinh} x = \log\left(x+\sqrt{x^2+1}\right)$

$y = \frac{e^x-e^{-x}}{2} \Rightarrow \dots \quad x = \log\left(y+\sqrt{y^2+1}\right)$