① $\sum \left(\frac{a^n}{n}+\frac{b^n}{n^2}\right)x^n \quad , a,b>0$

$$R=\frac{1}{\overline{\lim}\sqrt[n]{\frac{a^n}{n}+\frac{b^n}{n^2}}}$$

$M=\max\{a,b\}$

$\sqrt[n]{a^n+b^n}\to\max\{a,b\}$
$a,b>0$

$\sqrt[n]{n}\to 1$

$$\underset{M}{\underset{\downarrow}{M\leftarrow \frac{\sqrt[n]{a^n+b^n}}{(\sqrt[n]{n})^2}}} = \sqrt[n]{\frac{a^n+b^n}{n^2}} \le \underset{M\ (\text{То2П})}{\underset{\downarrow}{\sqrt[n]{\frac{a^n}{n}+\frac{b^n}{n^2}}}} \le \sqrt[n]{\frac{a^n+b^n}{n}} = \underset{M}{\underset{\downarrow}{\frac{\sqrt[n]{a^n+b^n}}{\sqrt[n]{n}}\to M}} \quad \Big/\lim_{n\to\infty}$$

$(\sqrt[n]{n})^2 \to 1$, $\sqrt[n]{n}\to 1$

$$\Rightarrow R=\frac{1}{M}$$

$\left(-\frac{1}{M},+\frac{1}{M}\right)$ ✓ $\qquad x=\pm\frac{1}{M}$?

$|x|>\frac{1}{M}$ ✗

1° $a>b$, $M=a$

$x=\frac{1}{a}$: $\sum\left(\frac{a^n}{n}+\frac{b^n}{n^2}\right)\left(\frac{1}{a}\right)^n=\sum\left(\frac{1}{n}+\Big(\underset{<1}{\frac{b}{a}}\Big)^n\cdot\frac{1}{n^2}\right) \longrightarrow$ див.

$\sum\left(\frac{b}{a}\right)^n\cdot\frac{1}{n^2}$, ПК: $\left(\frac{b}{a}\right)^n\cdot\frac{1}{n^2}\le\frac{1}{n^2}$ конв.

$\sum\frac{1}{n}$ див.

$x=-\frac{1}{a}$: $\sum\left(\frac{a^n}{n}+\frac{b^n}{n^2}\right)\left(-\frac{1}{a}\right)^n=\sum\left(\frac{(-1)^n}{n}+\underbrace{(-1)^n\left(\frac{b}{a}\right)^n\cdot\frac{1}{n^2}}_{\text{апс. конв.}}\right) \longrightarrow$ конв.

$\sum\frac{(-1)^n}{n}$ конв. (Лајбниц)

$D=\left[-\frac{1}{a},\frac{1}{a}\right)$

2° $a=b=M$

$x=\frac{1}{a}$: $\sum\left(\frac{a^n}{n}+\frac{a^n}{n^2}\right)\left(\frac{1}{a}\right)^n=\sum\left(\underset{\text{див.}}{\frac{1}{n}}+\underset{\text{конв.}}{\frac{1}{n^2}}\right) \longrightarrow$ див.

$\underline{x=\frac{1}{a}}$: $\sum\left(\frac{a^n}{n}+\frac{a^n}{n^2}\right)\left(\frac{1}{a}\right)^n=\sum\left(\underbrace{\frac{1}{n}}_{\text{див.}}+\underbrace{\frac{1}{n^2}}_{\text{конв.}}\right)$ → див.

$\underline{x=-\frac{1}{a}}$: $\sum\left(\frac{a^n}{n}+\frac{a^n}{n^2}\right)\left(-\frac{1}{a}\right)^n=\sum\left(\underbrace{\frac{(-1)^n}{n}}_{\text{конв.}}+\underbrace{\frac{(-1)^n}{n^2}}_{\text{апс.конв.}}\right)\longrightarrow$ конв.

$$D=\left[-\frac{1}{a},\frac{1}{a}\right)$$

3° $b>a$, $M=b$

$\underline{x=\frac{1}{b}}$: $\sum\left(\frac{a^n}{n}+\frac{b^n}{n^2}\right)\left(\frac{1}{b}\right)^n=\sum\left(\underbrace{\left(\underbrace{\frac{a}{b}}_{<1}\right)^n\cdot\frac{1}{n}}_{\text{конв.}}+\underbrace{\frac{1}{n^2}}_{\text{конв.}}\right)\longrightarrow$ конв.

ПК: $\left(\frac{a}{b}\right)^n\cdot\frac{1}{n}\leq\left(\frac{a}{b}\right)^n$ ✓

$\underline{x=-\frac{1}{b}}$: $\sum\left(\frac{a^n}{n}+\frac{b^n}{n^2}\right)\left(-\frac{1}{b}\right)^n=\sum\underline{(-1)^n}\cdot\left(\left(\frac{a}{b}\right)^n\cdot\frac{1}{n}+\frac{1}{n^2}\right)\longrightarrow$ апс.конв.

$$D=\left[-\frac{1}{b},\frac{1}{b}\right]$$

② Развити $\frac{1}{1-x^2}$ у степени ред око нуле.

$$\frac{1}{1-x^2}=(1\underbrace{-x^2}_{t})^{\overbrace{-1}^{\alpha}}=(1+t)^{\alpha}=\sum_{n=0}^{\infty}\binom{\alpha}{n}t^n=\sum_{n=0}^{\infty}\binom{-1}{n}(-x^2)^n=\sum_{n=0}^{\infty}\underbrace{(-1)^n\cdot(-1)^n}_{=1}\cdot x^{2n}=\sum_{n=0}^{\infty}x^{2n}$$

$t\in(-1,1)$

$-x^2\in(-1,1)\Rightarrow x^2\in[0,1)$

$\underline{x\in(-1,1)}$

$$\binom{-1}{n}=\frac{(-1)\cdot(-2)\cdot\ldots\cdot(-n)}{n!}=\frac{(-1)^n\cdot n!}{n!}=(-1)^n$$

$\boxed{Т}$ $f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n$ и R-полупр.конв. Тада:

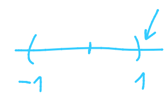
1) (непрекидност) f непр. на скупу $\{x \mid |x-x_0|<R\}$

2) (диференцијабилност) $f:(x_0-R,x_0+R)\to\mathbb{C}$ је ∞-пута диф. и важи:

$$f^{(k)}(x)=\sum_{n=0}^{\infty}\left(a_n(x-x_0)^n\right)^{(k)}=\sum_{n=0}^{\infty}a_n\cdot n\cdot(n-1)\cdot\ldots\cdot(n-k+1)\cdot(x-x_0)^{n-k}$$

„диференцирамо члан по члан"

$\boxed{\text{Т}}$ (Абел) $\sum a_n$ конв. и $f(x)=\sum_{n=0}^{\infty} a_n x^n$ и важи на $(-1,1)$.



Тада је $\lim_{x\to 1-} f(x) = \sum_{n=0}^{\infty} a_n = f(1)$

$\left[x \to \frac{x}{R} = t\right.$

③ Израчунати $\sum_{n=0}^{\infty} \frac{n}{2^n}$.

идеја: $n\cdot x^n,\ x=\frac{1}{2}$ $\quad \left(\sum x^n\right)' = \sum (x^n)' = \sum n\cdot x^{n-1}$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad /' \quad , x\in(-1,1)$$

$$\sum_{n=0}^{\infty} n\cdot x^{n-1} = -\frac{1}{(1-x)^2}(-1) = \frac{1}{(1-x)^2} \quad /\cdot x$$

$$\sum_{n=0}^{\infty} n\cdot x^n = \frac{x}{(1-x)^2} \quad \leftarrow x=\frac{1}{2} \quad \Rightarrow \sum_{n=0}^{\infty} n\cdot\left(\frac{1}{2}\right)^n = \frac{1/2}{(1/2)^2} = 2$$

④ Показати од $\sum_{n=0}^{\infty}(-1)^n\cdot x^{2n} = \frac{1}{1+x^2}$, доказати да је $\operatorname{arctg} x = \sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n+1}}{2n+1}$.

$$f(x) = \sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n+1}}{2n+1}, \quad R = \frac{1}{\overline{\lim}\sqrt[n]{|a_n|}} = \frac{1}{\overline{\lim}\sqrt[n]{\left|\frac{(-1)^n}{2n+1}\right|}} = \lim\sqrt[n]{2n+1} = 1$$

важи на $(-1,1)$ сигурно

$$\underset{(-1,1)}{f'(x)} = \sum_{n=0}^{\infty}(-1)^n\cdot\left(\frac{x^{2n+1}}{2n+1}\right)' = \sum_{n=0}^{\infty}(-1)^n\cdot x^{2n} = \frac{1}{1+x^2}$$

$$\Rightarrow f(x) = \int\frac{1}{1+x^2}dx = \operatorname{arctg} x + C$$

$C=?$ $\quad f(0) = \operatorname{arctg} 0 + C = C$

$f(0) = \sum_{n=0}^{\infty}(-1)^n\cdot\frac{0^{2n+1}}{2n+1} = 0$ $\Big\}$ $C=0 \Rightarrow f(x) = \operatorname{arctg} x.$

непр. на $(-1,1)$

(5) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = S$

$$\underline{\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2n+1} = \operatorname{arctg} x} \quad \text{на } (-1,1). \quad x=1?$$

Да ли $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ конв? Лајбниц $\frac{1}{2n+1} \searrow 0$ ✓

$$\Rightarrow \lim_{x\to 1-} \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^n}{2n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = S$$

$$\Rightarrow S = \lim_{x\to 1-} \operatorname{arctg} x = \operatorname{arctg} 1 = \frac{\pi}{4}.$$

(6) Развити у степене редове:

а) $\log(1+x+x^2+x^3)$

б) $\sin^3 x$

в) $\dfrac{1}{(1-x^2)\sqrt{1-x^2}}$

г) $\log(x+\sqrt{1+x^2})$

д) $x \arcsin x + \sqrt{1-x^2}$

ђ) $e^x \cdot \cos x$

а) $\log(1+\underbrace{x+x^2+x^3}_{t}) = \log(1+t) = \sum \frac{(-1)^n t^n}{n} = \sum \frac{(-1)^n}{n} \cdot (x+x^2+x^3)^n$ ✗ ?

$$\log(1+x+x^2+x^3) = \log((1+x)(1+x^2)) = \log(1+x) + \log(1+x^2) = \sum_{1}^{\infty} \frac{(-1)^{n+1} x^n}{n} + \sum_{1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{n} =$$

непарни, парни

$$= \sum_{n=1}^{\infty} \frac{(-1)^{2n} x^{2n-1}}{2n-1} + \sum_{n=1}^{\infty} \left(\frac{(-1)^{2n+1} \cdot x^{2n}}{2n} + \frac{(-1)^{n+1} \cdot x^{2n}}{n} \right) = \quad \text{на } (-1,1]$$

$$= \sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1} + \sum_{n=1}^{\infty} \left(\frac{-1 + (-1)^{n+1} \cdot 2}{2n} \right) \cdot x^{2n} = \sum_{n=1}^{\infty} a_n x^n$$

$x, x^2 \in (-1,1]$

$\Rightarrow \underline{\underline{x \in (-1,1]}}.$

$$a_n = \begin{cases} \frac{1}{n}, & 2 \nmid n \\ \frac{-1+2\cdot(-1)^{n/2+1}}{n}, & 2 \mid n \end{cases}$$

б) $\sin^3 x = \left(\sum \dots \right)^3$ ✗

$\sin 3x = \sin(2x+x) = \sin 2x \cos x + \cos 2x \sin x$

$= 2\sin x \cos^2 x + (\cos^2 x - \sin^2 x)\sin x$

$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$

$\sin 3x = 3\sin x - 4\sin^3 x$

$= 3\sin x - 4\sin^3 x$

$$\Rightarrow \sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x =$$

$$= \frac{3}{4}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} - \frac{1}{4}\sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!} = \dots = \frac{3}{4}\sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!}\left(1-9^n\right)$$

$$\left.\begin{matrix} x\in\mathbb{R} \\ 3x\in\mathbb{R}\end{matrix}\right\} \underline{\underline{x\in\mathbb{R}}}$$

в) $$\frac{1}{(1-x^2)\sqrt{1-x^2}} = \frac{1}{(1-x^2)^{3/2}} = (1-x^2)^{-3/2} = \sum_{n=0}^{\infty}\binom{-3/2}{n}(-x^2)^n = \frac{(2n+1)!!}{(2n)!!}\cdot x^{2n}$$

$$(-1)^n\cdot\binom{-3/2}{n} = (-1)^n\cdot\frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\dots\left(-\frac{2n+1}{2}\right)}{n!} = \underbrace{(-1)^n\cdot(-1)^n}_{1}\frac{\frac{(2n+1)!!}{2^n}}{n!} = \frac{(2n+1)!!}{2^n\cdot n!} = \frac{(2n+1)!!}{(2n)!!}$$

$$\boxed{(2n+1)!! = (2n+1)(2n-1)(2n-3)\cdot\dots\cdot 5\cdot 3\cdot 1}$$

$$2^n\cdot n! = (2\cdot 1)(2\cdot 2)(2\cdot 3)(2\cdot 4)\cdot\dots\cdot(2n) = 2\cdot 4\cdot 6\cdot\dots\cdot(2n) = (2n)!!$$

$$-x^2\in(-1,1) \Rightarrow$$
$$\underline{\underline{D=(-1,1)}}$$

г) $\log\left(x+\sqrt{1+x^2}\right) = f(x)$

на вид знаем "$\sum a x^n$"

$$f'(x) = \frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{1}{2\sqrt{1+x^2}}\cdot 2x\right) = \frac{1}{x+\sqrt{1+x^2}}\cdot\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}} = (1+x^2)^{-1/2} = \sum_{n=0}^{\infty}\binom{-1/2}{n}x^{2n} =$$

$$= \dots = \sum_{n=0}^{\infty}(-1)^n\cdot\frac{(2n-1)!!}{(2n)!!}x^{2n}, \quad |x|<1$$

→ отворен зф сме извод правили

„интегрално член по член"

уточнено: $g(x) = \sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{(2n)!!}\cdot\frac{x^{2n+1}}{2n+1}$, $R=1$ ← доказати $\Rightarrow g'(x) = \sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{(2n)!!}\cdot x^{2n}$, на $(-1,1)$

$$\Rightarrow f'(x) = g'(x) \quad \text{на } (-1,1)$$

$$\Rightarrow f(x) = g(x)+c$$

$x=0$: $\left.\begin{matrix} f(0) = \log\left(0+\sqrt{1+0^2}\right) = \log 1 = 0 \\ g(0) = \sum^{\infty} \dots 0^{2n+1} \end{matrix}\right\} c=0$

$x=0$: $f(0)=\log(0+\sqrt{1+0^2})=\log 1=0$

$g(0)=\sum_{n=0}^{\infty}\boxed{\cdots}\,0^{2n+1}=0$

$\Bigg\}\ c=0$

$$\Rightarrow f(x)=\log\left(x+\sqrt{1+x^2}\right)=g(x)=\sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{(2n)!!}\cdot\frac{x^{2n+1}}{2n+1}$$, сигурно важи на $(-1,1)$

да ли можемо и границе да укључимо?

Абел: може ако $\sum(-1)^n\frac{(2n-1)!!}{(2n)!!}\,\frac{(\pm1)^{2n+1}}{2n+1}$ конв!

испоставља се да $\sum(-1)^n\frac{(2n-1)!!}{(2n)!!}\,\frac{(\pm1)^{2n+1}}{2n+1}$ апсолутно конв.

домаћи

$\left(\sum\frac{(2n-1)!!}{(2n)!!}\cdot\frac{1}{2n+1}\leftarrow \text{конв}\right)$

По Абелу $\Rightarrow D=[-1,1]$.